Topology of Lie groups & loop groups. (Conan)

Ref: Bott Harvard lectures 1992. 2018 S.

§ Topology of compact Lie groups.

- G compact Lie group
- $\mathcal{L}G = \mathrm{Map}(S^1, G)$ loop group.

Eg. $SO(n) = \mathrm{Aut}(\mathbb{R}^n, g)^0$
$U(n) = \mathrm{Aut}(\mathbb{C}^n, g)$
$Sp(n) = \mathrm{Aut}(\mathbb{H}^n, g)$

$\underbrace{G_2,\ F_4,\ E_6,\ E_7,\ E_8}_{\text{exceptional Lie gp.}}$
$(\sim \mathbb{O})$

Up to product & finite covers, these are ALL of them.

- $m : G \times G \longrightarrow G$ group multiplication.

$\rightsquigarrow m^* : H^\bullet(G) \longrightarrow H^\bullet(G) \otimes H^\bullet(G)$

$$m^* x = 1 \otimes x + \sum_{\substack{i+j=\deg x \\ i,j \geq 1}} g_1^i \otimes g_2^j + x \otimes 1$$

$(\because\ g \longmapsto (e, g) \xrightarrow{m} g\ )$

Claim: $G \neq S^2$

Pf: Otherwise, $H^\bullet(S^2) = \mathbb{Q}\langle 1, x \rangle$ w/ $x \in H^2$
$m^*(1) = 1 \otimes 1$
$m^*(x) = 1 \otimes x + x \otimes 1$
Note $x^2 = 0$ $(\because \dim S^2 = 2)$
$0 = m^*(x^2) = (m^* x)^2$
$= (1 \otimes x + x \otimes 1)^2$
$= 2\, x \otimes x \neq 0$ $(\times)$

Similarly, $G \neq S^{2n}$

Extension of this argument by induction gives

<u>Theorem</u> (<u>Hopf</u>).

(1) $H^\bullet(G)_{\mathbb{Q}} \simeq H^\bullet(S^{2d_1+1} \times S^{2d_2+1} \times \cdots\cdots)$

(2) $H^\bullet(G)_{\mathbb{Q}} = \Lambda^\bullet \mathrm{Prim}(G)$

($\overset{\Psi}{x}$ primitive $\overset{\Delta}{\Longleftrightarrow}$ $m^* x = 1 \otimes x + x \otimes 1$)

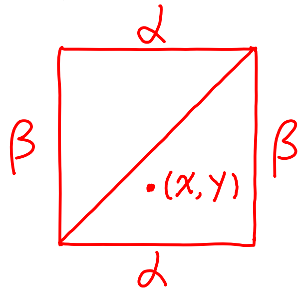
Adams: $G = S^n \Longleftrightarrow n = 1$ or 3.

i.e. $U(1) = SO(2)$ or $Sp(1) = SU(2) = Spin(3)$

• $\pi_1(G)$ Abelian.

reason: composition of 2 loops

= pointwise multi. of 2 loops in π_1



$$\xrightarrow{\alpha(x)\beta(y)} G$$

(diagonal)

$$\alpha * \beta \sim \alpha(x)\cdot\beta(x) \sim \beta * \alpha$$

$$U(n) \longrightarrow U(n+1)$$
$$\downarrow$$
$$S^{2n+1} \quad \underset{\text{up to dim. } 2n}{\sim} \quad pt.$$

$\underset{\pi_* F \to \pi_* E \to \pi_* B}{\Longrightarrow}$ $\pi_k(U(n)) \xrightarrow{\cong} \pi_k(U(n+1))$, $k < 2n$

$[-1]$

$\rightsquigarrow$ $\pi_k(U) \triangleq \lim\limits_{n\to\infty} \pi_k(U(n))$.

Same for $G = U(n), SO(n), Sp(n)$.

Cor $ST(n,k) := \dfrac{U(n+k)}{U(k)} \underset{\text{dim } s \ll k}{\overset{\text{up to}}{\sim}} pt$

- $$\begin{array}{ccc} U(n) & \longrightarrow & \dfrac{U(n+k)}{U(k)} \\ & & \downarrow \\ & & \dfrac{U(n+k)}{U(n)U(k)} = Gr_{\mathbb{C}}(n, n+k) \end{array}$$

This is an approximation to the universal $U(n)$-bundle, i.e. given any principal bundle, $U(n) \longrightarrow E \longrightarrow M^d$, $\exists$ k

$\exists !$ (up to homotopy) $f : M \longrightarrow Gr_{\mathbb{C}}(n, n+k)$

s.t.
$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U(n+k)/U(k) \arrow[d] \\ M \arrow[r, "f"'] & Gr_{\mathbb{C}}(n, n+k) \end{tikzcd}$$

Namely, $\{U(n)\text{-bdl}/M\}/\cong \;\overset{\sim}{\longleftrightarrow}\; [M, Gr_{\mathbb{C}}(n, n+k)]$

for $k \gg 0$

$$\begin{array}{ccc} U(n) \longrightarrow & \lim\limits_{k\to\infty} \dfrac{U(n+k)}{U(k)} =: EU(n) \overset{\text{h.e.}}{\sim} pt. \\ & \downarrow \\ & \lim\limits_{k\to\infty} Gr_{\mathbb{C}}(n, n+k) =: BU(n) \end{array}$$

Grassmannian of n-planes in Hilbert space

(($U(n)$-bdl.)) is representable by $BU(n)$.

i.e. $\{U(n)\text{-bdl.}/M\}/\cong \;\longleftrightarrow\; [M, BU(n)]$

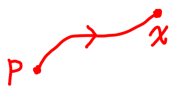
- $$G \longrightarrow \underset{\underset{*}{\wr}}{EG} \longrightarrow BG$$

$\Longrightarrow \quad \pi_\bullet(BG) = \pi_{\bullet-1}(G)$

(such de-looping is not easy in general.)

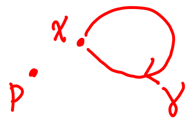
- $$\pi_\bullet(\Omega X) = \pi_{\bullet+1}(X) \quad \forall X$$

reason: $\Omega_p X \longrightarrow \underset{\underset{*}{\wr}}{P_p X} \xrightarrow{ev} X$ loop fibration

 $\longmapsto \quad x$

- $$\Omega_p X \longrightarrow \mathcal{L}X \longrightarrow X$$

(a Serre fib. $\Rightarrow$ spectral seq. ✓)

 $\longmapsto \quad x$

$\exists$ section (i.e. const. loops).

When $X = G$

$$\mathcal{L}G \xrightarrow{\sim} \Omega_e G \times G$$

$$\gamma(t) \longmapsto \left((\gamma(0))^{-1} \cdot \gamma(t)\ ,\ \gamma(0) \right)$$

$\Longrightarrow \quad \pi_\bullet(\mathcal{L}G) = \pi_{\bullet+1}(G) \oplus \pi_\bullet(G)$

G cpt $\Rightarrow \pi_2(G) = 0$ $\left(\pi_3(G) = \mathbb{Z} \text{ (if simple)}\right)$

If $\pi_1(G) = 0 \Rightarrow \pi_1(\mathcal{L}G) = 0$

$\qquad\qquad\qquad\quad\ \pi_2(\mathcal{L}G) = \pi_3(G) \neq 0$

$\Longrightarrow \exists$ (interesting) line bundle / $\mathcal{L}G$.

- $\mathcal{L}G$ is a group under pointwise multi.

Cor. $H^\bullet(\mathcal{L}G)$ is a Hopf algebra.

$$H^\bullet \xrightarrow{m^*} H^\bullet \otimes H^\bullet$$

If x primitive, i.e. $m^* x = 1 \otimes x + x \otimes 1$.

$x \in H^2 \Rightarrow x^k \neq 0,\ k > 0$ (okay $\because$ $\dim = \infty$)

$$\Longrightarrow \quad H^\bullet(\mathcal{L}G) = \Lambda^\bullet(\mathrm{Prim}) \otimes S^\bullet(\mathrm{Prim}(-1)).$$

§ Geometry of compact Lie groups.

$G \times G \longrightarrow G \leftrightarrow G \curvearrowright G$ left multi.

$\rightsquigarrow$ Adjoint action $G \curvearrowright G$

$$\mathrm{Ad}(g)h = g h g^{-1}$$

Ad-action fixes $e \in G$

$$\Rightarrow \mathrm{Ad} : G \curvearrowright T_e G = \{\text{left inv. vector fields on } G\}$$
$$= \mathfrak{g} := \Gamma(G, T_G)^{G_L}$$

$X, Y \in \Gamma(T_G)^{G_L} \Longrightarrow [X,Y] \in \Gamma(T_G)^{G_L}$

$\rightsquigarrow [\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ Lie algebra.

G compact $\xRightarrow{\int_G} \exists$ left & right inv. metric on G

$\Longrightarrow \exists$ Ad-inv. inner product on $\mathfrak{g}$

i.e. $\langle [X,Y], Z \rangle = \langle X, [Y,Z] \rangle$

$\Rightarrow G \curvearrowright \mathfrak{g}$ same as $G \curvearrowright \mathfrak{g}^*$

adjoint-action $\equiv$ coadjoint action

Eg. of adj-orbit.

$$G = SO(3) \curvearrowright \mathfrak{g} = \mathbb{R}^3 \qquad [X,Y] = X \times Y$$

Ad-action $\equiv$ usual rotations.

Ad-orbits : $S^2(r)$ or $\{0\}$

· Every $\underbrace{\text{line}}_{\text{Cartan subalg.}}$ meets $S^2(r)$ orthogonally at 2 pts.

• $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g}) \xRightarrow{\text{linearize}}$

$$\mathrm{ad} := d(\mathrm{Ad}) : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

Fact : $\mathrm{ad}\, x(y) = [x, y]$

• $x \in \mathfrak{g} \rightsquigarrow \mathfrak{g} \xrightarrow{\mathrm{ad}\, x} \mathfrak{g}$

$$\begin{array}{ccc} \mathfrak{g} & & \mathfrak{g} \\ \cup\!\!| & & \cup\!\!| \\ \underbrace{\mathrm{Ker}(\mathrm{ad}x)}_{\mathfrak{g}_x} & & \underbrace{\mathrm{Im}(\mathrm{ad}x)}_{\mathfrak{g}^x} \end{array}$$

denote $\mathfrak{g}_x$, $\mathfrak{g}^x$

Prop. $\mathfrak{g} = \mathfrak{g}_x \overset{\perp}{\oplus} \mathfrak{g}^x$

Pf. $\langle \mathfrak{g}_x, \mathfrak{g}^x \rangle \overset{\text{def}^{\text{n}} \text{ of } \mathfrak{g}^x}{=\!=} \langle \mathfrak{g}_x, [x, \mathfrak{g}] \rangle$

$\overset{\text{ad-inv. metric}}{=\!=} \langle [\mathfrak{g}_x, x], \mathfrak{g} \rangle$

$\overset{\text{def}^{\text{n}} \text{ of } \mathfrak{g}_x}{=\!=} 0$

dim count $\Longrightarrow$ Done

Def : $x \in \mathfrak{g}$ regular if $\dim \mathfrak{g}_x$ minimum.

call such a $\mathfrak{g}_x \leqslant \mathfrak{g}$ Cartan subalg.

($\dim \mathrm{Ker}$ is upper semi-cts. $\Rightarrow$ generic x is regular)

Prop. $\mathfrak{g}_x$ Cartan $\Longrightarrow$ Abelian

Pf. $z \in \mathfrak{g}_x$ (i.e. $[x, z] = 0$)

$\Rightarrow [x, x+tz] = 0 \xRightarrow{\text{Jacobi id.}} [ad_x, ad_{x+tz}] = 0$

$\Rightarrow ad_{x+tz}$ preserves decomp. $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}^x$

ev. for ad_x : 0 , $\neq 0$

$\Rightarrow$ For small t, $ad_{x+tz} : \mathfrak{g}^x \circlearrowleft$ non-sing.

$\Rightarrow \mathfrak{g}_{x+tz} \subseteq \mathfrak{g}_x$

$\Rightarrow \mathfrak{g}_{x+tz} = \mathfrak{g}_x$ ($\because x$ regular)

If $\mathfrak{g}_x$ NOT Abelian, i.e. $\exists\, y, z \in \mathfrak{g}_x$, $[y, z] \neq 0$

$\Rightarrow [y, x+tz] \neq 0$ ($\because [y, x] = 0$)

$\Rightarrow y \in \mathfrak{g}_x \setminus \mathfrak{g}_{x+tz}$ ($\longrightarrow\!\!\!\!\!\!\times\ $).

Ex. Cartan subalg. $\Longleftrightarrow$ max. Abelian subalg.

Let $\mathfrak{g} \supseteq \mathcal{O}_y := \{ g y g^{-1} \mid g \in G \}$ Ad-orbit.

$$f_x : \mathcal{O}_y \longrightarrow \mathbb{R}$$
$$f_x(z) = \langle x, z \rangle$$

(really a linear function on $\mathfrak{g}$)

Claim: $\mathrm{Crit}(f_x) = \mathcal{O}_y \cap \mathfrak{g}_x$.

(always $\perp$ intersections).

$z \in \mathrm{Crit}(f_x)$

$\Leftrightarrow 0 = \frac{d}{dt}\Big|_{t=0} \langle e^{tu} z e^{-tu}, x \rangle \quad \forall u \in \mathfrak{g}$

$= \langle [u, z], x \rangle = -\langle z, \underbrace{[u, x]}_{\mathfrak{g}^x} \rangle_{\forall u}$

$\Leftrightarrow z \perp \mathfrak{g}^x \Leftrightarrow z \in \mathfrak{g}_x$.

Cor: Any 2 Cartan subalg. are conjugate.

($\because$ any 2 elts in $\mathcal{O}_y$ are conj. to each other.)

Cor: Every $y \in \mathfrak{g}$ is conjugate to an elt. in $\mathfrak{g}_x$

[Conj. Thm] ($\because$ $\langle -, x \rangle$ must have a max. on $\mathcal{O}_y$)

Eg. $U(n) \xrightarrow[\text{Ad = Conjugat}^n]{} u(n) = \{$ skew-Herm. matrices $\}$

$\qquad\qquad \curvearrowright \; i\,u(n) = \{$ Herm. matrices $\}$

Choose $X = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \in i\,u(n)$

x regular $\Leftrightarrow$ λ_i's distinct $\Rightarrow$ $\mathfrak{g}_x = \text{Diag} \cap i\,u(n)$

Cor $\leftrightarrow$ any Hermitian matrix is diagonalizable.

(in fact, unique up to permutat^n ~ Weyl group)

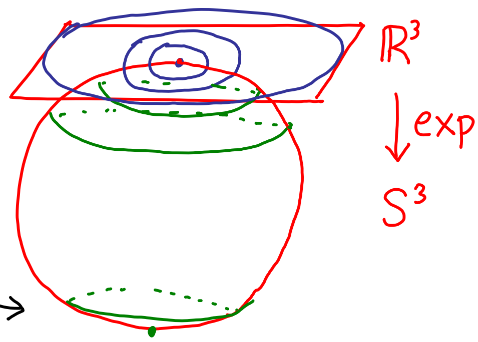
Coadj. orbit $Ad(G) \cdot X = G / G_x$ $\leftarrow$ stablizer

- (Fact) homog. complex manifold
- generically G/T

Topology is tractible.

Remark: ad-orbits in $\mathfrak{g}$ ~ Ad-orbits in G near e.

Eg. $SU(2) = S^3$



far away orbit will hit antipodal point.

Remark: <u>Noncompact</u> eg. $SL(2,\mathbb{R})$

$\exists$ adj. inv. non-degen. inner product on $\mathfrak{g}$

$\forall$ Lie alg. $\mathfrak{g} \rightsquigarrow$ Killing form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$

$$\kappa(x, y) := \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_x \circ \mathrm{ad}_y)$$

G semisimple $\overset{\text{def}^{\underline{n}}}{\iff}$ κ non-degenerate

Eg. $SL(n,\mathbb{R})$ semi-simple.

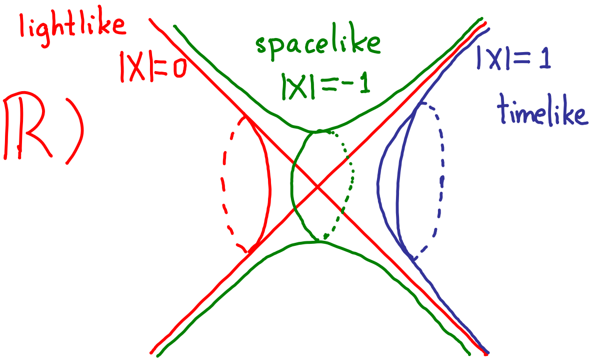
$$\mathfrak{sl}(2,\mathbb{R}) \ni X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$$\kappa(X, X) \equiv |X|^2 \underset{\text{(up to scalar)}}{=} -\det X = a^2 + bc$$

i.e. $(\mathfrak{sl}(2,\mathbb{R}), \kappa) \cong \mathbb{R}^{2,1}$

ad-orbit $\longleftrightarrow$ $\{ |X| = r \}$ hyperboloid.

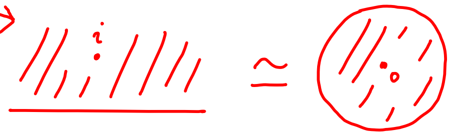
$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}^{2,1}$



No conj. thm. for $\mathfrak{sl}(2,\mathbb{R})$.

$\exists$ conj. thm. on timelike region ($\because f_X$ has max.)

Global picture of $SL(2,\mathbb{R})$

$$SL(2,\mathbb{R}) \xrightarrow{\frac{az+b}{cz+d}}$$  $\simeq$

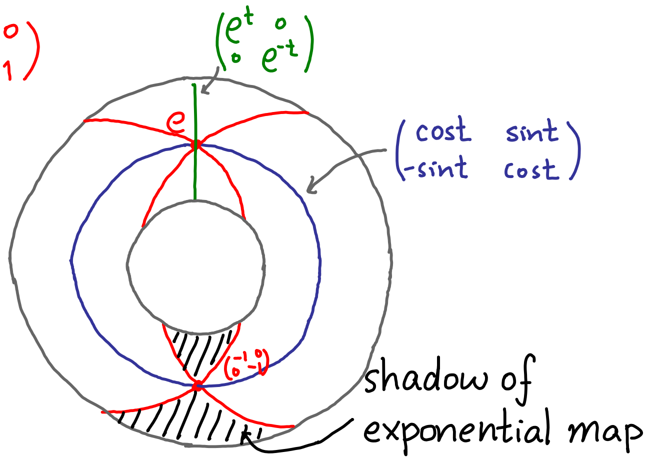
$$\begin{array}{l} SL(2,\mathbb{R}) \\ \quad \text{\rotatebox{90}{$\subseteq$}} \\ SO(2) = G_i \ \text{(isotropic subgp of } i\text{)} \end{array}$$

i.e. $\begin{array}{ccc} S^1 & \longrightarrow & SL(2,\mathbb{R}) \\ & & \downarrow \\ & & D^2 \end{array}$ $\left[\text{Compare: } \begin{array}{ccc} S^1 & \longrightarrow & SU(2) = S^3 \\ & & \downarrow \text{ Hopf fib.} \\ & & S^2 \end{array} \right.$

$$\Rightarrow \quad SL(2,\mathbb{R}) \simeq S^1 \times D^2$$

$SL(2,\mathbb{R}) \ni e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= S^1 \times D^2$



Ex. $A = \begin{pmatrix} -5 & * \\ 0 & -1/5 \end{pmatrix} \in SL(2,\mathbb{R})$ does not lie in the image of exponential map.

$\left[\begin{array}{l} \text{Hint: } \quad A = e^X \quad \exists X \in sl(2,\underset{\sim\sim}{\mathbb{C}}) \\ \quad \Rightarrow \quad X = \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \text{ w/ } \alpha \in \mathbb{R} \text{ or } i\mathbb{R} \\ \quad \Rightarrow \quad e^{tX} = \begin{pmatrix} e^{t\alpha} & * \\ & e^{-t\alpha} \end{pmatrix} \Rightarrow \operatorname{Tr} e^{tX} \in \mathbb{R}^+ \text{ or } S^1 \\ \quad \Rightarrow \quad \operatorname{Tr} e^{tX} \geqslant 2 \ \text{ or } \ \geqslant -2 \end{array} \right.$

Back to compact case.

Eg. $U(2)$ (rank $= 2$)

$$\underset{\Downarrow}{\underline{u}(2)} = \mathfrak{h} \oplus E$$

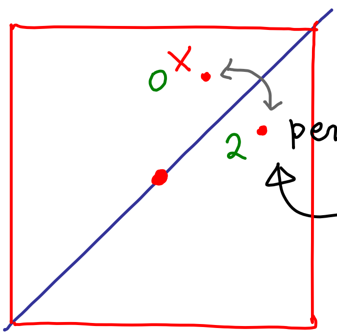
$$\begin{pmatrix} i\alpha_1 & z \\ -\bar{z} & i\alpha_2 \end{pmatrix} = \underbrace{\begin{pmatrix} i\alpha_1 & \\ & i\alpha_2 \end{pmatrix}}_{X} + \begin{pmatrix} & z \\ -\bar{z} & \end{pmatrix}$$

$z \in \mathbb{C}$ & $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\mathrm{ad}(X)\begin{pmatrix} & z \\ -\bar{z} & \end{pmatrix} = \left[i\begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}, \begin{pmatrix} & z \\ -\bar{z} & \end{pmatrix} \right] = \begin{pmatrix} 0 & i(\alpha_1-\alpha_2)z \\ -\overline{i(\alpha_1-\alpha_2)z} & 0 \end{pmatrix}$$

$\Longrightarrow$ $\mathfrak{g}_X = \mathfrak{h}$ unless $\alpha_1 = \alpha_2$

$\mathfrak{h}$



$\alpha_1 = \alpha_2$ (non-generic)

permute $\alpha_1 \leftrightarrow \alpha_2$

These 2 pts $= G\cdot X \cap \mathfrak{h}$
$= \mathrm{Crit}\ f_X$ of index 0 & 2.

General picture: $\mathfrak{h} := \mathfrak{g}_X$ Cartan

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

(1) $X \in \mathfrak{h}$ is regular unless X is in a system of hyperplanes in $\mathfrak{h}$.

(2) General $X \in \mathfrak{h}$ will decompose

$$\mathfrak{m} = E_1 \oplus \cdots\cdots \oplus E_j$$

on which $\mathrm{ad}(X)$ is given by $90°$ rotation $\times$ dilation.

Can choose complex structure on E s.t.
$\mathrm{ad}(X)|_E = i\,\alpha(X)\times$ $\exists\ \alpha \in \mathfrak{h}^*$ called roots

(e.g. $\alpha = \alpha_1 - \alpha_2$ for $SU(2)$).

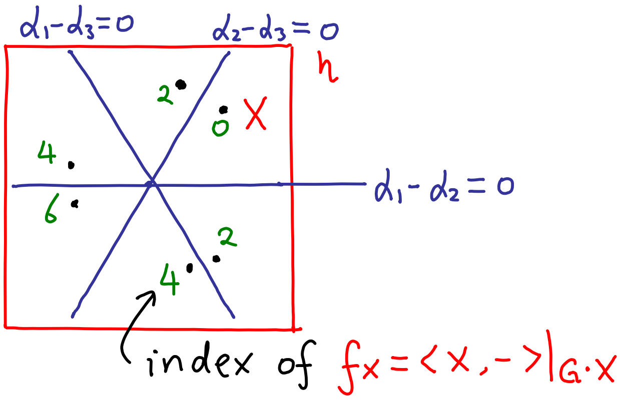
Eg. $\mathfrak{u}(n) = \mathfrak{h} \oplus \bigoplus_{i<j} E_{ij}$

$$\mathfrak{h} \ni i\begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix}, \qquad E_{ij} \ni \begin{pmatrix} \ddots & & z & \\ & & & \\ -\bar{z} & & \ddots & \\ & & & \end{pmatrix}$$

roots: $\sqrt{-1}\,(\alpha_i - \alpha_j)$'s.

Eg. $SU(3)$

$\mathfrak{h} = \{\alpha_1 + \alpha_2 + \alpha_3 = 0\}$
$\simeq \mathbb{R}^2$



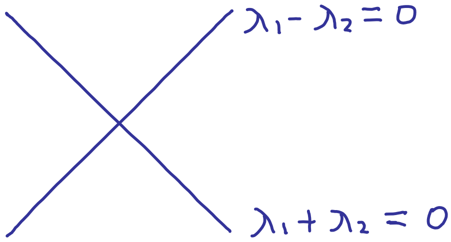
Index $f_X \in 2\mathbb{Z}$

$\xRightarrow{\text{Morse}}$ $H^\bullet(\underbrace{G\cdot X}; \mathbb{Z})$ has no torsion.

$P_t(SU(3)/T) = 1 + 2t^2 + 2t^4 + t^6.$

Eg. $SO(4)$ (rk 2, dim 6)

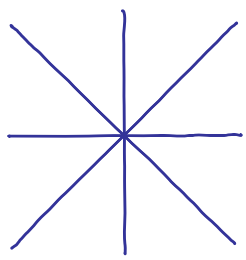
$$\mathfrak{h} \ni X = \left(\begin{array}{cc|cc} & \lambda_1 & & \\ -\lambda_1 & & \multicolumn{2}{c}{0} \\ \hline & & & \lambda_2 \\ \multicolumn{2}{c|}{0} & -\lambda_2 & \end{array}\right) \in \left\{\left(\begin{array}{cc|cc} & \lambda_1 & & \\ -\lambda_1 & & \multicolumn{2}{c}{*} \\ \hline & & & \lambda_2 \\ \multicolumn{2}{c|}{*} & -\lambda_2 & \end{array}\right)\right\} = \mathfrak{g}$$



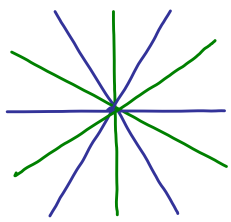
(look like $SU(2)\times SU(2)$ locally)

Eg. $SO(5)$ (rk 2, dim 10)

$$\mathfrak{h} \ni X = \left(\begin{array}{cc|cc|c} & \lambda_1 & & & \\ -\lambda_1 & & \multicolumn{2}{c|}{0} & 0 \\ \hline & & & \lambda_2 & \\ \multicolumn{2}{c|}{0} & -\lambda_2 & & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 \end{array}\right) \in \left\{\left(\begin{array}{cc|cc|c} & \lambda_1 & & & \\ -\lambda_1 & & \multicolumn{2}{c|}{*} & * \\ \hline & & & \lambda_2 & \\ \multicolumn{2}{c|}{*} & -\lambda_2 & & * \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & 0 \end{array}\right)\right\} = \mathfrak{g}$$



Eg. Last rank 2 diagram: G_2



§ Review of Morse theory

$f : M \longrightarrow \mathbb{R}$ Morse

Morse cpx. $C_k \triangleq \bigoplus_{\mathrm{ind}(p)=k} \mathbb{R}\langle p \rangle \xrightarrow{\partial} C_{k-1}$

count grad. flow lines

Thm: $H_\bullet(C_\bullet, \partial) \cong H_\bullet^{\mathrm{sing}}(M, \mathbb{R})$

$$\mu_t(f) := \sum \#\left\{\begin{matrix}\text{index } k \\ \text{crit. pt.}\end{matrix}\right\} t^k$$

$$P_t(M) := \sum \dim H_k \, t^k \quad \text{Poincaré polyn.}$$

Thm. $\Longrightarrow$ 1) $\mu_t(f) - P_t(M) = (1+t)\, Q(t)$

$\exists \; Q(t) = a_0 + a_1 t + a_2 t^2 + \cdots\cdots$

w/ $a_j \geq 0 \quad \forall j$

2) $Q(t) = 0 \iff \partial = 0$

$\Longrightarrow \mathrm{Tor}\, H_\bullet(M, \mathbb{Z}) = 0$

called perfect Morse function.

3) $\mathrm{index}\, f \in 2\mathbb{Z} \Longrightarrow \partial = 0$

<u>Completion Principle</u>

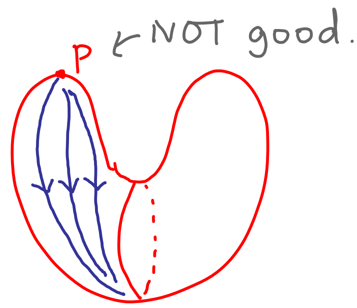
If $\forall \; p \in \mathrm{Crit}(f)$,

the unstable submfd. of p

can be extended to a cycle, say N_p,

then N_p's is a basis for $H_\bullet(M, \mathbb{Z})$

and $\nexists$ torsion.



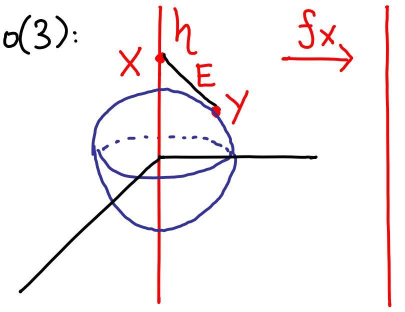
Back to adjoint orbit

$\mathcal{O}(y_0) = Ad(G)\cdot y_0 \subseteq \mathfrak{g}^*$

Recall: regular $X \in \mathfrak{g}$

$\rightsquigarrow \mathfrak{g} = \underbrace{\mathfrak{h}}_{\mathfrak{g}_X} \oplus E_{\alpha_1} \oplus E_{\alpha_2} \oplus \cdots \oplus E_{\alpha_m}$

so(3):



$f_X(Y) = \langle X, Y\rangle|_{\mathcal{O}(y_0)}$

$E(Y) = \mathrm{dist}^2(X, Y)$

$= |X - Y|^2$

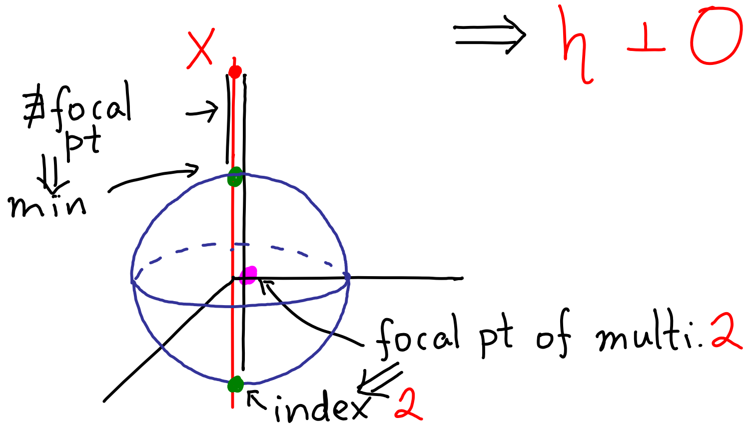
$= \underbrace{|X|^2}_{\text{fixed}} - 2\underbrace{\langle X, Y\rangle}_{f_X(Y)} + \underbrace{|Y|^2}_{\text{const. on } \mathcal{O}(y_0)}$

i.e. $E|_{\mathcal{O}(y)}$ has same critical points (& index) as $f_X|_{\mathcal{O}(y)}$

In particular, at $p \in \mathrm{Crit}(f_X)$ on $\mathcal{O}(Y)$

$\overline{pX} \perp \mathcal{O}(Y)$ at p ($\because E$ is dist. fu.)

$\Rightarrow \mathfrak{h} \perp \mathcal{O}(Y)$ along $\mathrm{Crit}(f_X)$.

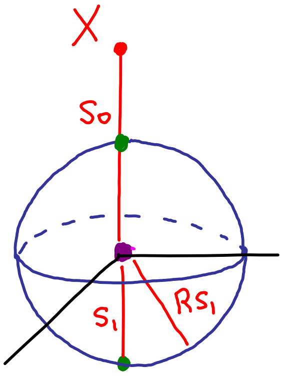


Recall: Focal point of $N \subset M$

$TM|_N = TN \oplus \nu_N$ ← normal bdl.

$\exp: \nu_N \longrightarrow M$ (of same dim!)

focal set $\stackrel{\text{def.}}{=}$ crit. values of $\exp$.



$R \in SO(3)$ rotation about 0

$\rightsquigarrow$ new path $s_0 + Rs_1$,
w/ same length as $s_0 + s_1$.

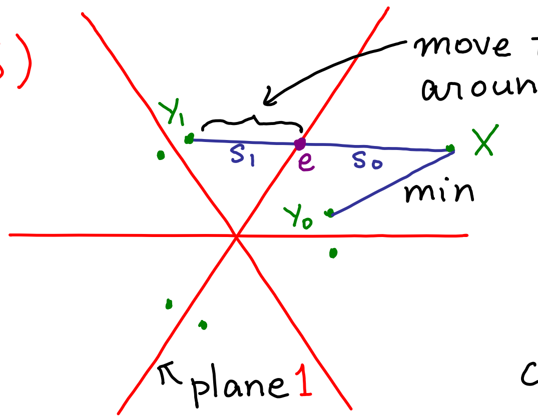
$\rightsquigarrow$ S^2 $(= SO(3)/S^1)$ family of paths from X to $O(Y)$ of same length.

$$S^2 \hookrightarrow \Omega_{X \to S^2}\mathfrak{g} \qquad (\mathfrak{g} \simeq \mathbb{R}^3)$$

Claim: Deform $\Rightarrow$ completion cycle for the critical point.

- $\Omega_{X\to S^2}\mathbb{R}^3 \overset{\text{h.e.}}{\simeq} S^2$ by deforming to straight lines.

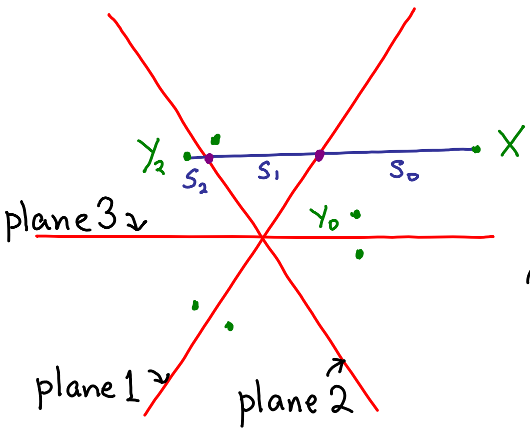
Eg. $SU(3)$



$s_0 + s_1$
$\wr$
$s_0 + Ad(g_1)s_1$

$g_1 \in P_1 \subset SU(3)$
centralizer of plane 1

$$\rightsquigarrow S^2 = P_1/T \hookrightarrow O(Y_0)$$



Can rotate wrt 2 points.

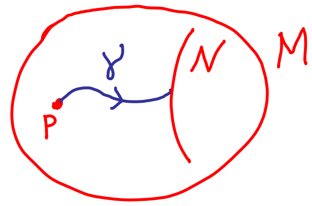
$s_0 + Ad(g_1)s_1 + Ad(g_1)Ad(g_2)s_2$

$$\rightsquigarrow \underbrace{P_1 \times P_2/T^2}_{S^2 \tilde{\times} S^2 \,:\, S^2\text{-bdl}/S^2} \longrightarrow O(Y_0)$$

- max. point $\rightsquigarrow$ $\underset{\text{tower of } S^2}{\widetilde{\prod} S^2} \xrightarrow{\deg 1} G/T$ Bott-Samelson variety
- $H^\bullet(G/T)$ is direct summand of $H^\bullet(\widetilde{\prod} S^2)$

§ Morse theory on loop spaces

Geodesics. $p \notin N \subset (M, g)$



Energy $E : \Omega_{p \to N} M \longrightarrow \mathbb{R}$

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot\gamma|^2 \, dt$$

- If $M = \mathbb{R}^n \Rightarrow \Omega_{p\to N} M \overset{\text{h.e.}}{\simeq} N$
  (deforming to straight lines).

- Crit. point of $E \longleftrightarrow$ geodesic

  Write $X = \dot\gamma$, $Y = \delta\gamma$ variation v.f.

$$\begin{aligned}
\delta_Y E &= \frac{1}{2}\int_0^1 Y \langle X, X \rangle \, dt \qquad (\because \nabla g = 0) \\
&= \frac{1}{2}\int_0^1 \left(\langle \nabla_Y X, X \rangle + \langle X, \nabla_Y X \rangle \right) dt \\
&= \int_0^1 \langle \underbrace{\nabla_Y X}_{\nabla_X Y + \cancel{[Y, X]}}, X \rangle \, dt
\end{aligned}$$

come from push forward $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.
(and $\mathrm{Tor}\nabla = 0$)

$$\begin{aligned}
&= \int_0^1 \left(X \langle Y, X \rangle - \langle Y, \nabla_X X \rangle \right) dt \\
&= \langle Y, X \rangle \Big|_p^N - \int_0^1 \langle Y, \nabla_X X \rangle \, dt
\end{aligned}$$

EL eqt. $\delta_Y E = 0 \ \forall Y \iff \dot\gamma(1) \perp N$ & $\nabla_X X = 0$.

i.e. geodesic

- Morse theory for $E = \int_\Sigma |du|^2 : \mathrm{Map}(\Sigma^d, M) \to \mathbb{R}$

$d =$	1	2	≥ 3
	✓	"compensated" bubbling (conformal)	✗

(reason: Sobolev inequality).

• Thm. A : If $\nexists$ geodesic γ w/ $E(\gamma) \in (a,b)$,

then $\Omega^{<b} \underset{h.e.}{\simeq} \Omega^{<a} := \{\gamma \in \Omega_{p\to N} M : E(\gamma) < a\}$

• Thm. B : If $\exists !$ geodesic γ w/ $E(\gamma) \in (a,b)$,

If γ is non-degenerate w/ $\text{index}(\gamma) = \lambda$

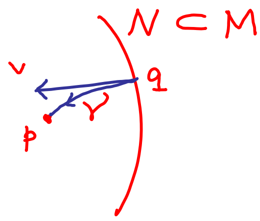
then $\Omega^b \sim \Omega^a \cup e^{\lambda}$ ⟵ λ dim. cell

• A + B ⇒ Can use finite dim. construction.

• (Equivalent) def$^{\underline{n}}$ of non-degen. & index:

$\exp : \nu_{N/M} \longrightarrow M$

If $(q, v) \longmapsto p$



⇝ geodesic γ from q to p, i.e. $\gamma(t) = \exp(q, tv)$

γ non-degen $\Leftrightarrow$ $d\exp(q,v) : T_{(q,\nu)} \nu_{N/M} \xrightarrow{\cong} T_p M$

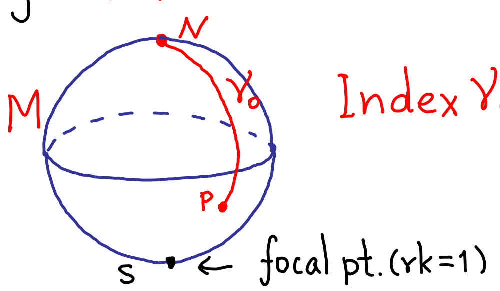
p focal point of N along γ of rank k

$\Leftrightarrow$ $k = \dim \operatorname{Ker} \left(d\exp(q,v) : T_{(q,\nu)} \nu_{N/M} \longrightarrow T_p M \right)$

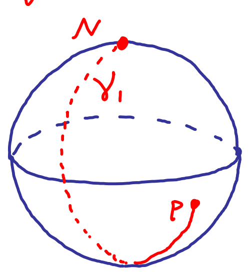
If γ non-degen., then

index $\gamma \triangleq$ # focal points along γ

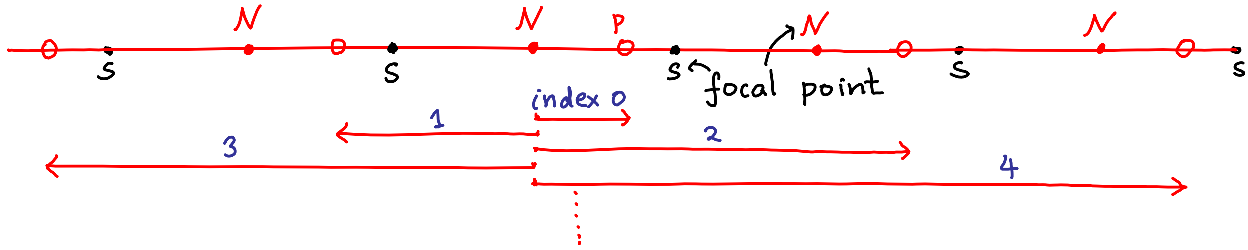
Eg. $M = S^2 \supset N = \{\text{north pole}\}$



Index $\gamma_0 = 0$



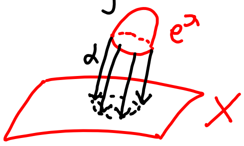
Index $\gamma_1 = 1$.

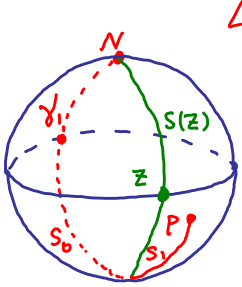


$$\xRightarrow{\text{Morse}} \quad \underbrace{\Omega_{P\to N} S^2}_{\Omega S^2} \sim e_0 \cup e_1 \cup e_2 \cup e_3 \cup e_4 \cup \cdots\cdots$$

What are the attaching maps?

(Recall attaching cell $X \cup_{\alpha} e^n = X \sqcup e^n / \alpha(x) \sim x$ w/ $x \in \partial e^n$
$\alpha : \partial e^n \to X$)





$\gamma_1 = S_0 + S_1$

$\forall \; z \in S^1_{\text{equator}} \rightsquigarrow$ (piecewise smooth)

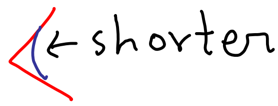
$\varphi(z) = S(z) + S_1$ path from N to S

$$\text{length}(\varphi(z)) = \text{length}(\gamma_1)$$

$\varphi(z)$ has a corner, unless $\varphi(z) = \gamma_1$

Smooth out corner, $\varphi(z)$

$\rightsquigarrow \tilde{\varphi}(z)$ shorter!



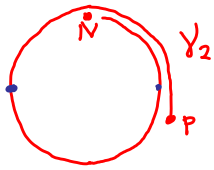
$$\rightsquigarrow \quad \tilde{\varphi} : S^1_{\text{equator}} \longrightarrow \Omega^{\leq E(\gamma_1)}$$

has a unique max length (at $\gamma = \varphi(z) = \tilde{\varphi}(z)$)

i.e. $\tilde{\varphi}$ 'links' this critical pt./geodesic.

$\rightsquigarrow$ elt. in $H_1(\Omega S^2)$.

For next geodesic.



Similarly, $\tilde{\varphi} : S^1 \times S^1 \longrightarrow \Omega^{< E(\gamma_2) - \varepsilon}$

$\tilde{\varphi}$ perturbs $\varphi(z_1, z_2) = S^-(z_1) + S^+(z_2) + S_2$

$\rightsquigarrow$ torus 'links' a 2-cell.

Repeat $\rightsquigarrow$ basis of $H_\bullet(\Omega S^2)$

Remark: Easier for ΩS^n w/ $n \geq 3$ since

$$\Omega S^n \sim pt \cup e_1^{n-1} \cup e_2^{2(n-1)} \cup \cdots$$

must be zero in H. ∂e_2 (could be non-trivial in π.)

Second variations: (w/ $\perp_N$ at bdy).

$$\delta_Y^2 E = -\int_0^1 \langle \nabla_Y Y, \cancel{\nabla_X X} \rangle - \int_0^1 \langle Y, \underbrace{\nabla_Y \nabla_X X}_{\nabla_X\nabla_Y - \cancel{\nabla_{[X,Y]}} - R(X,Y)} \rangle$$

at crit. pt.

$$= -\int \langle Y, \nabla_X \nabla_Y X \rangle + \int \langle Y, R(X,Y)X \rangle$$

$$= \int \langle Y, (-\nabla_X^2 + R(X,-)X)Y \rangle \, dt$$

Jacobi equation $L(Y) \equiv -\nabla_X^2 Y + R(X,Y)X = 0$

Y: Jacobi field along geodesic γ.

$\dim \{$Jacobi fields along $\gamma\} = 2 \cdot \dim M$

- \# neg. eigenvalues of L on $\Gamma(TM|_\gamma)_{(BC)}$
  = \# directions of steepest descents of E at γ.

Boundary condition (BC): $\begin{cases} Y(t=0) = 0 \\ \nabla_X Y + S_X Y = 0 \text{ at } t=1. \end{cases}$

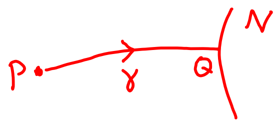
$$0 = Y\langle Y, X\rangle|_N$$

$$= \underbrace{\langle \nabla_Y Y, X\rangle|_N}_{\text{tensorial in } Y \;=:\; \langle S_X Y, Y\rangle} + \langle Y, \nabla_Y X\rangle|_N$$

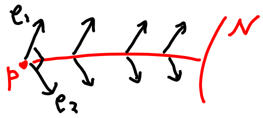
$$\left(\langle \nabla_{fY} fY, X\rangle|_N = f(Q)\big(\langle (\nabla_Y f)\cancel{Y}, X\rangle + f\langle \nabla_Y Y, X\rangle\big) = f^2 \langle \nabla_Y Y, X\rangle|_N \right)$$

$(\because \dot\gamma \perp N)$

Shape operator: $S_X : T_Q N \circlearrowleft$ via $\langle S_X Y, Y\rangle \stackrel{\Delta}{=} \langle \nabla_Y Y, X\rangle(Q)$



Choose orthormal frame at p, parallel translate along $\gamma \rightsquigarrow e_i$'s



Jacobi eqt. for $Y(t) = \sum_i x_i(t)\, e_i$ $\exists$ fu. $x_i(t)$'s

Write $\vec{X} = (x_1(t), \ldots, x_n(t))^t$

$$-\vec{X}'' + R(t)\vec{X} = 0, \quad R(t) \text{ } n\times n \text{ matrix}$$

w/ bdy condition $\vec{X}(0) = 0$; $\dot{\vec{X}}(1) + S\vec{X}(1) = 0$

<u>Thm</u>: Eigenvalues $(-\vec{X}'' + R\vec{X} = \lambda \vec{X})$ are discrete and bounded from below.

<u>Morse</u>: 1° # negative e.v.
= # focal pt. of N along γ

2° = $\text{index}_\gamma E$ on any permissable "polygonal approximation."

($\because \forall$ geodesic γ w/ $\ell(\gamma) < \varepsilon_M$ is unique (& abs. min) w/ fixed boundary points.)

Back to G:

- generic $X \in \mathfrak{g} \Rightarrow Q = e^X \in T \subset G$ $\exists!$ max torus T
- Indeed $T = e^{\mathfrak{g}_X}$. Write $\mathfrak{h} = \mathfrak{g}_X$.
- $|X|$ suff small $\Rightarrow$ all geodesics from e to Q lie in T

$$T \leqslant G \overset{Ad}{\curvearrowright} \mathfrak{g} \underset{\text{as } T\text{-mod}}{=\!=} \mathfrak{h} \oplus E_1 \oplus E_2 \oplus \cdots\cdots \oplus E_m$$

$T \curvearrowright \mathfrak{h}$ trivial

$T \curvearrowright E_i$ via orthogonal transf

$\overset{\text{choose}}{\underset{\text{ori.}}{\rightsquigarrow}}$ $E_i \simeq \mathbb{C} \curvearrowleft T \xrightarrow{\alpha_i} S^1 \subseteq \mathbb{C}^\times$

(reverse ori. $\rightsquigarrow \alpha_i^{-1}$)

roots: $\alpha_i^{\pm 1}$'s $\in \mathrm{Hom}(T, S^1) = T^*$

$$\begin{array}{ccc} \mathfrak{h} & \supset \pi^{-1}(e) \quad , & \pi^{-1}(\operatorname{Ker}\alpha_i) \\ {\scriptstyle \pi := \exp}\downarrow & \text{lattice} & \text{affine hyperplanes.} \\ T & & \end{array}$$

$\rightsquigarrow$ Diagram in $\mathfrak{h}$

Eg. $SO(3)$ $\qquad \mathfrak{g} = \mathfrak{h} \oplus E_1$

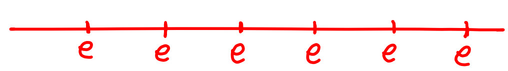
Claim: $\alpha_1 : T \xrightarrow{\cong} S^1$

$$T \ni \tilde{R}_\theta = \left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right) =: \left(\begin{array}{c|c} R_\theta & 0 \\ \hline 0 & 1 \end{array}\right)$$

$$E_1 \ni E = \left(\begin{array}{c|c} 0 & b \\ \hline -b^* & 0 \end{array}\right)$$

$$Ad(\tilde{R}_\theta)\cdot E = \left(\begin{array}{c|c} 0 & R_\theta b \\ \hline -(R_\theta b)^* & 0 \end{array}\right)$$

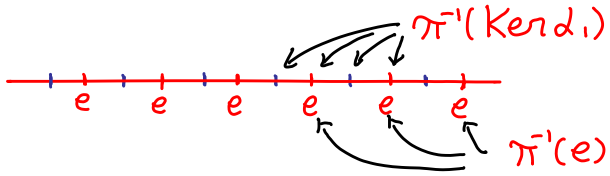
$\Rightarrow$ Diagram



Eg. $SU(2) \geqslant T \ni \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, $E = \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \in E_1$

$$\mathrm{Ad}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \cdot E = \begin{pmatrix} 0 & z e^{2i\theta} \\ -\bar{z} e^{-2i\theta} & 0 \end{pmatrix}$$

$\Longrightarrow \alpha_1 : T \longrightarrow S^1$ deg 2 cover

$\rightsquigarrow$ 

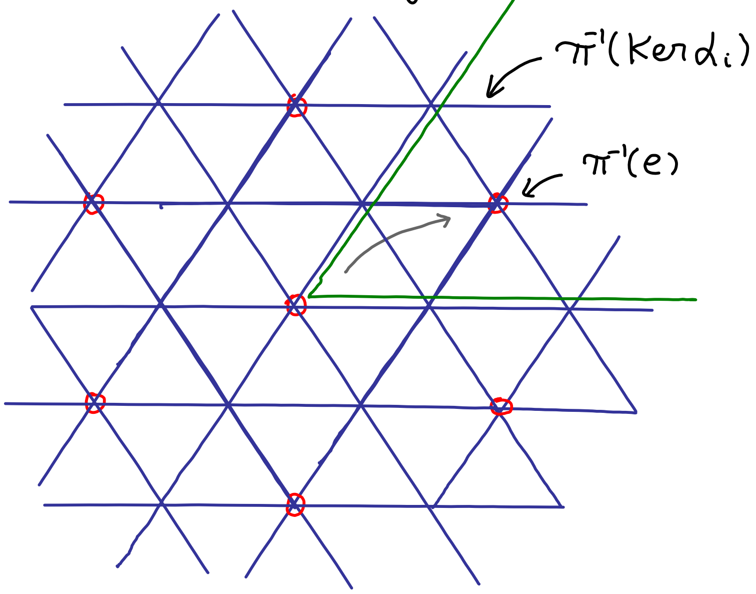
walls

wt. lattice

• $\mathrm{Ad} : SU(2) \longrightarrow SO(3)$, every torus/circle in $SU(2)$ is a double cover of a torus in $SO(3)$.

Eg. $SU(3)$

$\mathfrak{h}$



Prop: G semisimple, $\pi_1 = 0$

$\Longrightarrow$ 1) lattice $\pi^{-1}(e) \leqslant \mathfrak{h}$ is given by reflections of e wrt root planes.
(i.e. $\pi^{-1}(\mathrm{Ker}\,\alpha_i)$'s)

2) Lattice formed by $\bigcap$(root planes) $= \pi^{-1}(e)$ for $\mathrm{Ad}(G)$

Eg. $\pi_1(SU(n)) = 0$, $\mathrm{Ad}(SU(n)) = SU(n)/\overbrace{\mathbb{Z}_n}^{\text{center}}$
$= PU(n)$

• Return to geodesics on G:

Lemma: Geodesic $\gamma \subset (M,g)$, $X := \dot{\gamma}$

Y infinitesimal isometry, i.e. $\mathcal{L}_Y g = 0$

$\Longrightarrow$ $\langle X, Y \rangle$ is const. along γ.

Pf. $X\langle X,Y\rangle = \langle \overset{\text{geodesic}}{\cancel{\nabla_X X}}, Y\rangle + \langle X, \nabla_X Y\rangle$

$\overset{\mathrm{Tor}\nabla=0}{=} \langle X, \nabla_Y X - [Y,X]\rangle$

$= \frac{1}{2} Y \underbrace{\langle X, X\rangle}_{\text{const}} - \underbrace{\langle X, \mathcal{L}_Y X\rangle}_{=0} \quad \because \mathcal{L}_Y$ skew-symm.

Cor. Any geodesic through $\{Y=0\} \subset M$ is perpendicular to Y everywhere.

Cor. $G \longrightarrow \mathrm{Isom}(M) \curvearrowright M$, then

$\forall$ geodesic $\gamma \subset M$ w/ $\gamma(0) \in M^G$

$\Longrightarrow$ $\gamma \perp G$-orbits.

Note: $e \in G^{\mathrm{Ad}(G)}$

generic $Q \in T$

$\Longrightarrow$ (1) $\forall$ geodesic γ from e to Q, $\gamma \subset T$

(2) (Ex). $\mathrm{Ad}(G)\cdot Q$ is orthogonal complement to T in G.

Weyl group $W := \overset{\text{normalizer}}{N(T)}/T$

i.e. autom. of T induced from inner autom of G.

<u>Theorem</u> W is generated by reflections wrt root planes through $0 \in \mathfrak{h}$.

- A description of the diagram:

1° Choose fundamental domain $\mathcal{F}$ for $W \curvearrowright \mathfrak{h}$

2° For root plane of $\mathcal{F}$

$\rightsquigarrow \quad \alpha : \mathfrak{h} \longrightarrow \mathbb{R}$ character.

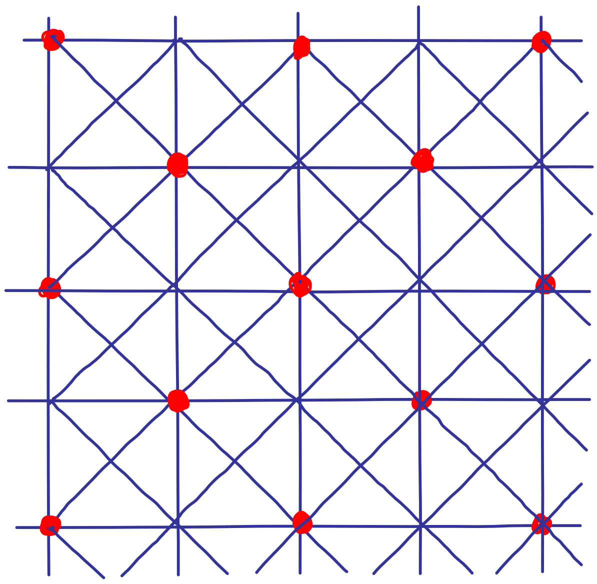
$\alpha(\exp(h)) = e^{2\pi i \alpha(h)}$

Planes $\sim \alpha^{-1}(\mathbb{Z})$ Orient α s.t. $\alpha|_{\mathcal{F}} > 0$

Simple roots $\sim$ those α's corresp. to $\partial\mathcal{F}$.

Cartan matrix $\left(\alpha_i(\Lambda_j)\right)_{\ell\times\ell}$

Eg. Diagram of $Spin(5)$:



$$\frac{\text{Blue lattice}}{\text{Red lattice}} \quad \pi^{-1}(0)$$

$= \mathbb{Z}_2$

($\simeq C(Spin(5))$ center).

Study geodesics in G from p to e:

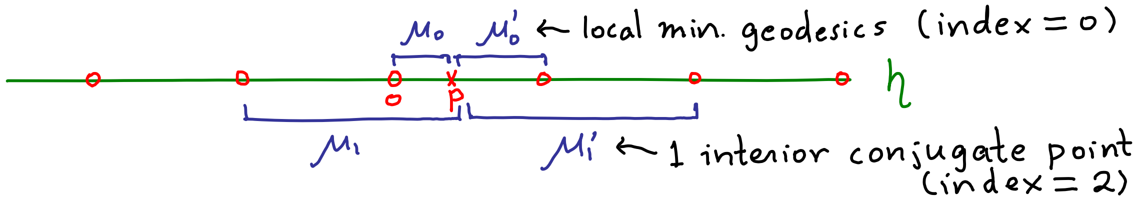
Choose $G \supset T \ni p$

Assume p generic $\Rightarrow$ $T = Z_G(p)$ centralizer

$\Rightarrow$ all such geodesics $\subset T$ ($\because$ totally geodesic)

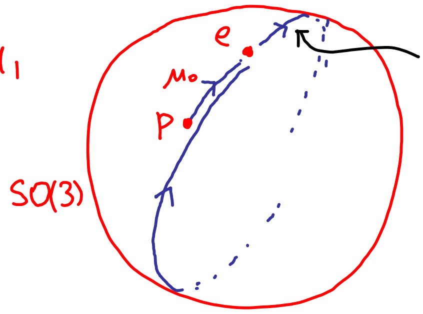
$\Rightarrow$ can be easily described in $\mathfrak{h}$

Eg. $SO(3) \simeq \mathbb{RP}^3 \rightsquigarrow \mathfrak{h} \simeq \mathbb{R}$



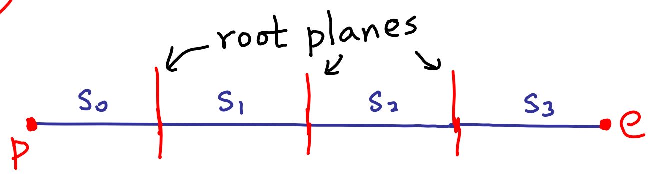
$\mu_0 + \mu_0' \sim \pi_0(\Omega_{p\to e} SO(3)) \simeq \pi_1(SO(3)) \simeq \mathbb{Z}_2$

μ_1



rotates by $SO(3)$ (up to stab $= S^1$)

$\rightsquigarrow$ 2 dim. link.



$S = S_0 + S_1 + S_2 + S_3 + S_4$

geodesic segment in G from p to e in diagram

$P_1 :=$ Stabilizer subgp. of end pt. of s_0

$\mathfrak{h} \oplus E_{\alpha_1} \rightsquigarrow S^3 \times T^{n-1}$ $(n = \mathrm{rk}\, G)$

$P_2 :=$ ——— " ——— s_1

$P_3 :=$ ——— " ——— s_2

$\rightsquigarrow \mu_s : P_1 \times P_2 \times P_3 \longrightarrow \Omega_{p\to e} G$

$\mu_s(x_1, x_2, x_3)$

☆ $= s_0 + x_1 s_1 x_1^{-1} + x_1 x_2 s_2 x_2^{-1} x_1^{-1} + x_1 x_2 x_3 s_3 x_3^{-1} x_2^{-1} x_1^{-1}$

Check: well-defined.

• $t \in T \subset P_i \quad \forall\, i$

$$\mu_s(x_1 t,\ t^{-1}x_2,\ x_3) = \mu_s(x_1,\ x_2,\ x_3)$$
$$= \mu_s(x_1,\ x_2 t,\ t^{-1}x_3)$$
$$= \mu_s(x_1,\ x_2,\ x_3 t)$$

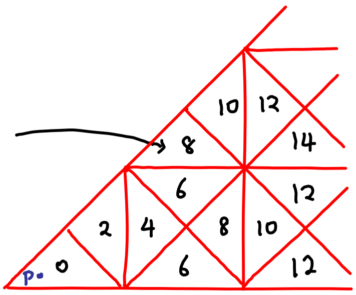
$$\Longrightarrow \quad T^3 \overset{\text{free}}{\curvearrowright} P_1 \times P_2 \times P_3 \xrightarrow[T^3\text{-inv.}]{\mu_s} \Omega_{p\to e} G$$

i.e. $\mu(x_1 t_1,\ t_1^{-1} x_2 t_2,\ t_2^{-1} x_3 t_3) = \mu(x_1, x_2, x_3)$

$$\rightsquigarrow \quad \underbrace{P_1 \times_T P_2 \times_T P_3 / T}_{V(1,2,3).} \xrightarrow{\mu_s} \Omega_{p\to e} G$$

Eg. Spin(5)

each simplex gives 1 geodesic (if $\pi_1 G = 0$) w/ these indexes.



Fundmental chamber

Given G w/ $\pi_1 = 0$, $\forall$ simplex Δ

$\rightsquigarrow$ 1 geodesic S_Δ (i.e. critical pt. of E)

$$\mu_\Delta : V_\Delta \longrightarrow \Omega_{p\to e} G$$

$\dim V_\Delta = 2 \times \#$planes crossed going from Δ to 0.

($=$ index S_Δ)

(deform $V_\Delta \rightsquigarrow$) linking mfd. of $S_\Delta \in \Omega_{p\to e} G$.

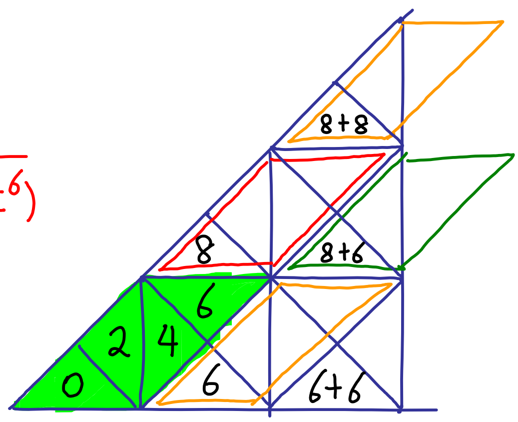
$\Longrightarrow H_\cdot(\{\mu_\Delta : V_\Delta \to \Omega_{pe} G\})$'s $\sim$ base of $H_\cdot(\Omega_{pe} G, \mathbb{Z})$

Eg. $P_t(\Omega_{pe}\, Spin(5))$

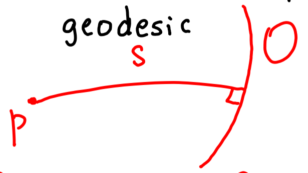
$$= \frac{1 + t^2 + t^4 + t^6}{(1 - t^6)(1 - t^8)} = \frac{1}{(1-t^2)(1-t^6)}$$

matchs w/ $Spin(5) \simeq_{\mathbb{Q}} S^3 \times S^7$

(via Serre spectral sequence).

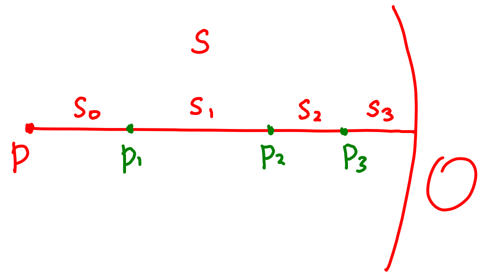


Note: compact $G \curvearrowright M \ni p$ generic point
$ \cup$
$ \mathcal{O}$: G-orbit



G_p (resp. G_s) : stabilizer of p (resp. s)

- p generic $\Longrightarrow$ $\dim G_p = \dim G_s$
- $\dim G_p > \dim G_s \Longrightarrow$ p is focal point of $\mathcal{O}$ along s.



s geodesic
$\rightsquigarrow$ p_i's s.t. $\dim G_{p_i} > \dim S$

$S = s_0 + s_1 + s_2 + s_3$

$$\rightsquigarrow \underbrace{G_{p_1} \underset{G_s}{\times} G_{p_2} \underset{G_s}{\times} G_{p_3}/G_s}_{V_s} \xrightarrow{\mu_s} \Omega_{p\mathcal{O}} M$$

where $(G_s)^3 \curvearrowright G_{p_1} \times G_{p_2} \times G_{p_3}$ free action.

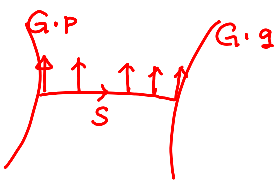
$$(x_1, x_2, x_3) \cdot (g_1, g_2, g_3) = (g_1 x_1, x_1^{-1} g_2 x_2, x_2^{-1} g_3 x_3)$$

Claim: μ_s can be deformed slightly, so that $E \circ \mu_s$ has an isolated non-degen. max. at (e,e,e).

Def: $G \curvearrowright M$ Variationally complete
(if every focal point arise from G action)

if Y solves $-\nabla_X^2 Y + R(X,Y)X = 0$ along any geodesic s,

s.t. Y(end pts) $\in T.$(orbits)



$\Longrightarrow Y \in \mathfrak{g}|_s$

Theorem $G \curvearrowright M$ variationally complete

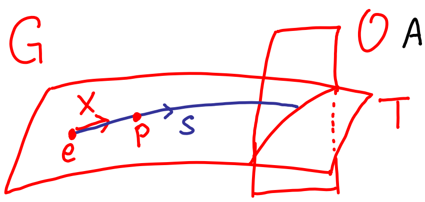
$\Longrightarrow \mu_s[V_s]$'s forms base for $H_\bullet(\Omega_{p\to O}M)$

- In general, V_s not orientable, $\rightsquigarrow$ base $/\mathbb{Z}_2$.
  If V_s's orientable, then $\rightsquigarrow$ base $/\mathbb{Z}$.

- If $G=\{e\}$ $\quad p \xrightarrow{s} O=\{q\}$
  tangent to orbit $\equiv$ vanishing
  variationally complete $\Longleftrightarrow$ $\nexists$ conjugate point
  e.g. $K_M \leqslant 0$
  $\nexists$ conj. pt. in M along any geodesic
  $\Longrightarrow$ $\Omega_{p\to q}M \sim$ Set of geodesics joining p to q.

Prop. $G \overset{Ad}{\curvearrowright} G$ is variationally complete

Pf: $G \supset s$ geodesic thru. a generic point P
Choose max. torus $T \ni P$ (generic in T)



$\Rightarrow s(t) = Pe^{tX}$ $\quad \exists X \in \mathfrak{h} = \mathrm{Lie}\,T$

Linearization of isometry action always give Jacobi fields.

$G_L \curvearrowright G \curvearrowleft G_R$ isometry ($\because$ bi-inv. metric on G)

$\Rightarrow \mathfrak{g}_L \oplus \mathfrak{g}_R \longrightarrow J(s) = \{\text{Jacobi field along } s\}$

$\dim J(s) \underset{q}{=} 2\dim G.$ ($\because 2^{nd}$ order ODE)

$(Y_L, Y_R) \longmapsto Y_L P e^{tX} + P e^{tX} Y_R$

However $\mathfrak{h}_L$ & $\mathfrak{h}_R$ give same Jacobi fields.

$(\because\ Y \in \mathfrak{h} \Rightarrow Y P e^{tX} = P e^{tX} Y$ as $X \in \mathfrak{h})$

On a (flat) torus T, both $Y P e^{tX}$ and $t P e^{tX}$ are Jacobi fields. Hence

$(\mathfrak{g} = \mathfrak{h} \oplus E)$

$$\psi : \mathfrak{h} \oplus \mathfrak{h} \oplus E \oplus E \xrightarrow{\cong} J(s)$$

$$\psi(Y_1, Y_2, E_1, E_2) = Y_1 P e^{tX} + t Y_2 P e^{tX} + E_1 P e^{tX} + P e^{tX} E_2$$

Suppose $Y = \psi(Y_1, Y_2, E_1, E_2)$ is tangent to 2 Adjoint orbits along s at t_1 and $t_2\ (\neq t_1)$.

Want $Y \in \mathfrak{g}|_s$.

Recall for adj. orbit $\mathcal{O} \subset \mathfrak{g}$, $L_g^{-1}(T_g\mathcal{O}) = (I - Ad(g))\mathfrak{g} \subseteq T_eG$

So, after left translating back to $e \in G$ $\quad (g_i = P e^{t_i X})$

$$\begin{aligned} Y_1 + t_1 Y_2 + Ad(g_1^{-1})E_1 + E_2 &= Z_1 - Ad(g_1^{-1})Z_1 \\ \underbrace{Y_1 + t_2 Y_2}_{\in \mathfrak{h}} + \underbrace{Ad(g_2^{-1})E_1 + E_2 = Z_2 - Ad(g_2^{-1})Z_2}_{\in E} \end{aligned} \qquad \exists Z_1, Z_2 \in E$$

$$t_1 \neq t_2 \implies Y_1 = Y_2 = 0$$

Also $\quad Ad(Pe^{tX})^{-1}E_1 + E_2 = Z - Ad(Pe^{tX})^{-1}Z \quad \forall t$

i.e. $\quad Y(t)$ tangent to Adj. orbit $\forall t$

i.e. $\quad Y \in \mathfrak{g}|_s$. $\qquad$ QED.

In general, for any compact symmetric G/K

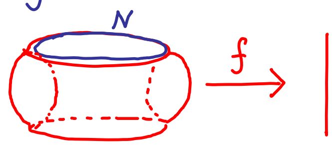
$K \curvearrowright G/K$ is variationally complete.

eg. Adj. action $G \curvearrowright G \equiv \left(G_\Delta \curvearrowright \dfrac{G \times G}{G_\Delta}\right)$

$SO(n) \curvearrowright S^n = SO(n+1)/SO(n)$

§ Non-degenerate critical manifold

$$N \subset M \xrightarrow{f} \mathbb{R}$$

eg. T^2 

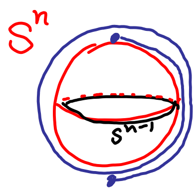
(i) N manifold, (ii) $df|_N = 0$

(iii) $\mathrm{Hess}(f)$, as a quadratic of normal bdl. $\nu_{N/M}$

$$\mathrm{Null}(\mathrm{Hess}(f)) = T_N$$

Thm. B becomes $M_b = M_a \cup_{\partial} \nu_{N/M}$.

As we cross a crit. mfd N, add cell of $\dim \geq rk(\nu^-_{N/M})$

Eg. $\Omega_{N\to S} S^n = S^{n-1} \cup \underbrace{\nu^-_{S^{n-1}}}_{\substack{\lambda = 2(n-1) \\ \text{2 copies of } S^{n-1}}} \cup \underbrace{\nu^-_{S^{n-1}}}_{\lambda = 4(n-1)} \cup \cdots$ 

$$= S^{n-1} \cup e_{2(n-1)} \cup \text{higher cell.}$$

$\Rightarrow \forall k < 2(n-1)-2$

$$\underbrace{\pi_k(\Omega S^n)}_{\pi_{k+1}(S^n)} \simeq \pi_k(S^{n-1})$$

$\rightsquigarrow \quad \pi_k^s(S^0) \triangleq \pi_{n+k}(S^n)$ for $n \gg 0$

Eg. $p = I$ and $q = -I \in SU(2n)$

$$\Omega_{pq} SU(2n) \ni \gamma : [0,\pi] \longrightarrow SU(2n)$$

$$\gamma(\theta) = \left(\begin{array}{c|c} e^{i\theta} I_n & 0 \\ \hline 0 & e^{-i\theta} I_n \end{array}\right)$$

need even dim

γ is an absolute min. for E.

Critical submfd. containing $\gamma \simeq \dfrac{U(2n)}{U(n)\times U(n)}$ (via conjugation).

$$\Rightarrow \Omega SU(2n) = \frac{U(2n)}{U(n)U(n)} \cup \cdots\cdots$$

Write $\gamma \leftrightarrow 1^n\ (-1)^n$

Next critical point:

$\leftrightarrow 1^n\ \ 1\ (-1)^{n-2}\ (-3)$ i.e. $\theta \mapsto \begin{pmatrix} e^{i\theta} I_n & & & \\ & e^{i\theta} & & \\ & & e^{-i\theta} I_{n-2} & \\ & & & e^{-3i\theta} \end{pmatrix}$

Corresp. critical mfd $= \dfrac{U(2n)}{U(n+1)\,U(n-2)\,U(1)}$

$$\Longrightarrow \Omega SU(2n) = \frac{U(2n)}{U(n)\,U(n)} \cup e_{2n-5} \cup \cdots\cdots$$

Letting $n \longrightarrow \infty$

$$\Omega SU = \frac{U}{U \times U}$$

$$\Longrightarrow \underbrace{\pi_k(\Omega SU)}_{\pi_{k+1}(SU)} = \pi_k\left(\frac{U}{U\,U}\right)$$

Similar for $\Omega \dfrac{U(2n)}{U(n)U(n)} = SU(n) \cup$ higher cell

(okay for loop space of symmetric spaces.)

$\Longrightarrow SU \overset{\Omega}{\underset{\Omega}{\rightleftarrows}} \dfrac{U}{U\,U}$ Bott periodicity.

$\rightsquigarrow$ K-theory for complex vector bundles.

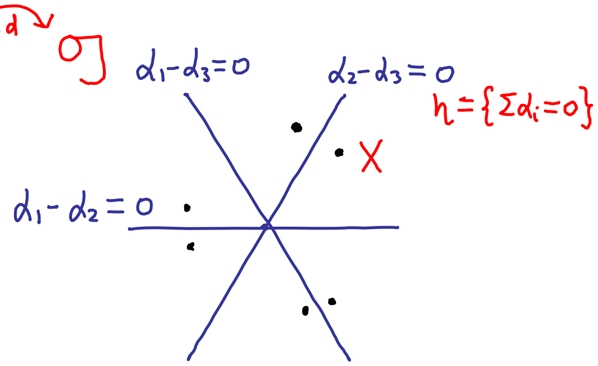
$$\begin{aligned}
\Omega(SO) &= SO/U && \cup \cdots\cdots \\
\Omega(SO/U) &= U/Sp && \cup \cdots\cdots \\
\Omega(U/Sp) &= Sp/Sp\,Sp && \cup \cdots\cdots \\
\Omega(Sp/Sp\,Sp) &= Sp && \cup \cdots\cdots \\
\Omega(Sp) &= Sp/U && \cup \cdots\cdots \\
\Omega(Sp/U) &= U/O && \cup \cdots\cdots \\
\Omega(U/O) &= O/O\,O && \cup \cdots\cdots \\
\Omega(O/O\,O) &= SO && \cup \cdots\cdots
\end{aligned}$$

(use $J = \oplus\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\rightsquigarrow$ complex str.)

$\rightsquigarrow$ 8-fold periodicity

$\sim$ KO-theory for real vector bundles.

Review: Ad-orbit $G \overset{Ad}{\curvearrowright} \mathfrak{g}$



Eg. $SU(3)$

$$X = \begin{pmatrix} i\alpha_1 & & \\ & i\alpha_2 & \\ & & i\alpha_3 \end{pmatrix} \in \mathfrak{h}$$

$$\alpha_1+\alpha_2+\alpha_3=0$$

$$\text{orbit} = \{ UXU^{-1} : U \in SU(3) \}$$

If X general $\alpha_i \neq \alpha_j \quad \forall i,j$

$$\Rightarrow \quad Ad(G)\cdot X \simeq \frac{U(3)}{U(1)U(1)U(1)}$$

If X s.t. $\alpha_1 = \alpha_2 \neq \alpha_3$

$$\Rightarrow \quad Ad(G)\cdot X \simeq \frac{U(3)}{U(2)U(1)} \qquad \text{etc.}$$

For $SU(n)$, adjoint orbits:

$U(n)/U(k_1)U(k_2)\cdots U(k_r)$ w/ $n=\sum_{j=1}^{r} k_j$ Flag varieties.

(Partial) Flag varieties

$Fl = \{$ filtrations of $\mathbb{C}^n$ by subsp $0 \underset{k_1}{\subsetneq} A_1 \underset{k_2}{\subsetneq} A_2 \underset{k_3}{\subset} \mathbb{C}^n \}$ ← codim

- Homogeneous space of $GL(n,\mathbb{C})$
  Complex manifold

$$Fl = GL(n,\mathbb{C}) \Big/ \left\{ \begin{pmatrix} \boxed{*} & * & * \\ & \boxed{*} & * \\ 0 & & \boxed{*} \end{pmatrix} \right\} = U(n) \Big/ \left\{ \begin{pmatrix} \boxed{*} & & 0 \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{pmatrix} \right\}$$

$$= \frac{U(n)}{U(k_1)U(k_2)U(k_3)}$$

- Eg. Complex Grassmannian $Gr(r,n)$.

Coadj. orbit $M \triangleq \mathrm{Ad}(G)\cdot X \qquad X \in \mathfrak{h} \subset \mathfrak{g}$

$\simeq G/L, \quad T \leqslant L \leqslant G$ closed subgp.

$(= G/T$ if $X \in \mathfrak{h}$ generic)

$$E : M \subset \mathfrak{h} \xrightarrow{\text{linear}} \mathbb{R}$$

Critical points of E on M is a Weyl orbit.

Unstable mfds $\rightsquigarrow$ cell decomposition of M.

In algebraic geometry, this can be obtained without Morse theory, called Bruhat decomposition.

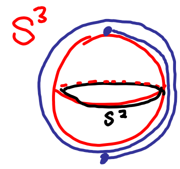
e.g. $$\begin{array}{ccc} \mathbb{CP}^1 & = & GL(2,\mathbb{C}) / \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \\ \cup & & \cup \\ \mathbb{C} & = & \left\{ \left[\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right] \right\} \end{array}$$ affine cell.

Based loop groups $\Omega_\cdot G$

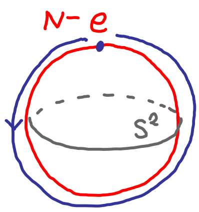
Eg. $G = SU(2) \cong S^3 \quad (S^2 = G/T)$

recall $\Omega_{N\to S} S^3 = S^2 \cup \underbrace{\nu^-_{S^2}}_{\lambda = 2\times 2} \cup \underbrace{\nu^-_{S^2}}_{\lambda = 4\times 2} \cup \cdots$

2 copies of S^2



$\Omega_\cdot G = \Omega_{N\to N} S^3 = pt. \cup \underbrace{\nu^-_{S^2}}_{\lambda=2} \cup \underbrace{\nu^-_{S^2}}_{\lambda=4} \cup \underbrace{\nu^-_{S^2}}_{\lambda=6} \cup \cdots$



$$= G/T \cup \begin{array}{c} E_1 \\ \downarrow D^2 \\ G/T \end{array} \cup \begin{array}{c} E_2 \\ \downarrow D^4 \\ G/T \end{array} \cup \begin{array}{c} E_3 \\ \downarrow D^6 \\ G/T \end{array} \cup \cdots\cdots$$

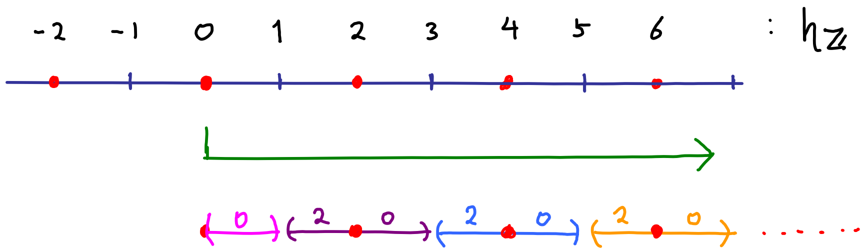
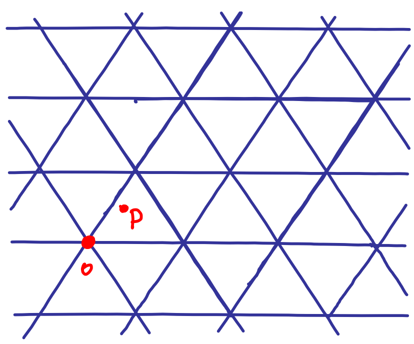


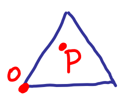
Diagram for $G/T = S^2$

This diagramatic description of $\Omega SU(2)$ generalizes to $\Omega_\cdot G$ for any cpt. gp. G.

Eg. $\Omega_{e\to p} SU(3)$



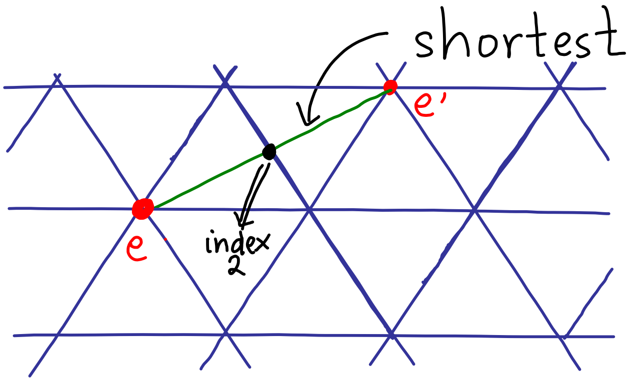
$\hat{W}$: group generated by reflections wrt these affine hyperplanes.

$\hat{W}$ acts on above 'diagram' w/ fund. domain 

If $p = e$, then critical pt. in $\Omega.G$

$\longleftrightarrow$ closed geodesic in G thru e

$\longleftrightarrow$ 1 parameter subgp. ($\Rightarrow$ smooth even at e)

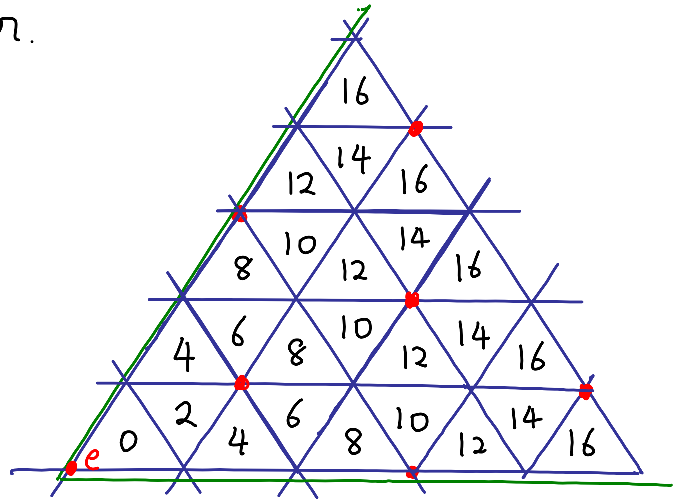


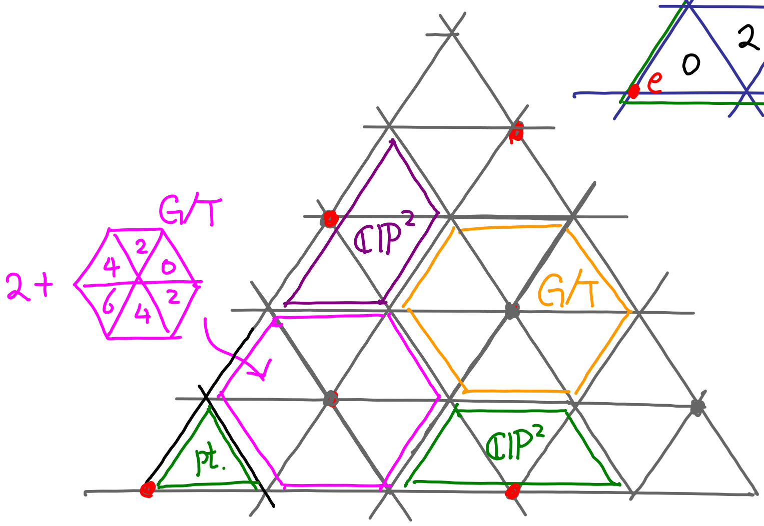
$\Rightarrow$

$\Omega.G = pt \cup E_1 \cup$ higher cell.

where E : D^2-bdl / G/T

Critical sets are labelled by lattice points • is a fixed chamber.





$$\Omega.G = pt. \cup \underset{\substack{\downarrow D^2 \\ G/T}}{E_1} \cup \underset{\substack{\downarrow D^8 \\ \mathbb{CP}^2}}{E_2} \cup \cdots\cdots$$

Attaching maps are difficult to describe.

Free loop groups

$\mathcal{L}G \triangleq \mathrm{Map}(S^1, G)$ (ptwise multi $\Rightarrow$ gp str.)

$\mathcal{L}G \twoheadrightarrow \Omega G$ via $\gamma(t) \longmapsto \gamma(t)\gamma^{-1}(1)$

$\rightsquigarrow 1 \longrightarrow \underset{\substack{\shortparallel \\ \{\text{const. loops}\}}}{G} \longrightarrow \mathcal{L}G \longrightarrow \Omega G \longrightarrow 1$

Claim: $\exists$ central extension

$$1 \longrightarrow S^1 \longrightarrow \widehat{\mathcal{L}G} \longrightarrow \mathcal{L}G \longrightarrow 1$$

s.t. ΩG is a coadj. orbit !

(so it has a cell decomposition.)

View $\mathcal{L}G$ as $\mathcal{G}$ for trivial G-bdl $/S^1$

§ Connections & curvature.

Principal G-bdl. $G \longrightarrow P \longrightarrow M$

($\Leftrightarrow$ $P \curvearrowleft G$ free action (s.t. $M = P/G$))

P trivial $\Longleftrightarrow$ P has a section

(G-torsor T, pick $s \in T \rightsquigarrow \underset{s}{T} \underset{\longmapsto}{\xrightarrow{\sim}} \underset{e}{G}$)

Group of gauge transformations:

$$\mathcal{G} = \mathrm{Aut}(P) \ni h : \begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & P \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd}$$ and $h(p \cdot g) = h(p) \cdot g$

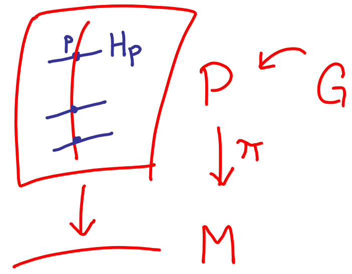
If $P = M \times G \Longrightarrow \mathcal{G} = \mathrm{Map}(M, G)$

If $P = S^1 \times G \Longrightarrow \mathcal{G} = \mathcal{L}G$

Space of connections:

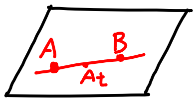
$\mathcal{A} = \mathcal{A}(P) \ni A$

Recall: Connection is an G-equivariant splitting of



$$0 \longrightarrow T_{vert}P \longrightarrow TP \longrightarrow \pi^* TM \longrightarrow 0$$

- $\mathcal{A}$ an affine space



(i.e. $A, B \in \mathcal{A} \Rightarrow A_t = tA + (1-t)B \in \mathcal{A} \ \forall t \in \mathbb{R}$)

If A & B 2 splitting, then

$$A - B \in \Gamma(P, \pi^* T^*_M \otimes T_{vert}P)$$

G-equivariant $\Rightarrow$ descend to M

$$A - B \in \Omega^1(M, \mathrm{ad}P)$$

where $\mathrm{ad}P = P \times_G \mathfrak{g}$ (Ad) VB/M

$$\mathcal{A} = A + \Omega^1(M, \mathrm{ad}P)$$

- Differential form viewpoint.

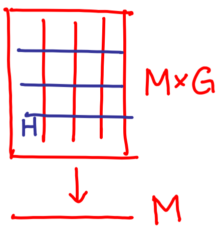
$$T_pP \supseteq H_p = \mathrm{Ker}\,\Theta_{A,p}$$

w/ $\Theta_A \in \Omega^1(P, \mathfrak{g})^G$ (right Ad)

- $\exists$ canonical $\Theta = g^{-1}dg \in \Omega^1(G, \mathfrak{g})^G$

Namely conn. on G-bdl. / 1 point.

By pullback, this is the trivial conn. on $P = M \times G$:



Θ?

$$g : P = G \longrightarrow G \quad \text{(The identity map)}$$

$$\Theta : T_gP \xrightarrow{dg} T_gG \xrightarrow{L_g} T_eG = \mathfrak{g}$$

i.e. $\Theta = g^{-1}dg \in \Omega^1(P, \mathfrak{g})^G$ (right Ad)

When $P = M \times G$ trivial

$\rightsquigarrow \exists$ canon. connection Θ on $\mathcal{A}$

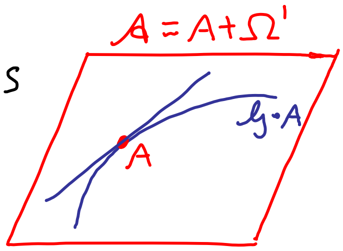
$$\Longrightarrow \quad \Omega^1(M,\mathfrak{g}) \xrightarrow{\sim} \mathcal{A}$$
$$A \longmapsto \operatorname{Ker}\Theta_A = H_A$$

with $\Theta_A = g^{-1}dg + g^{-1}\pi^*A\,g$

- $\mathcal{G} \curvearrowright \mathcal{A} \ni A$

$$\mathrm{Lie}\,\mathcal{G} = \Omega^0(M,\mathrm{ad}P), \quad T_A\mathcal{A} = \Omega^1(M,\mathrm{ad}P)$$

The linearization of action at A is



$$\Omega^0(M,\mathrm{ad}P) \xrightarrow{D_A} \Omega^1(M,\mathrm{ad}P)$$

(see below) $\left(D_A\xi = d\xi + [A,\xi] \right.$ wrt loc. trivializat$^{\underline{n}}$)

$\operatorname{Ker} D_A =$ infinitesimal stabilizer of A

- $P = M \times G \Longrightarrow \mathcal{G} = \mathrm{Map}(M,G)$

$$\mathcal{G} \curvearrowright \mathcal{A} \xrightarrow{\sim} \Omega^1(M,\mathfrak{g}) \ni A$$
$$h \cdot A = h^{-1}dh + h^{-1}Ah$$

Write $h_t = e^{t\xi}$ w/ $\xi \in \Omega^0(M,\mathfrak{g})$,

$$\left.\frac{d}{dt}\right|_{t=0}(h_t \cdot A) = d\xi + A\xi - \xi A = D_A\xi$$

Now $\underline{M = S^1}$ and $P = M \times S^1$ $\Longrightarrow$ $\mathcal{G} \curvearrowright \mathcal{A}$, with $\mathcal{G} = \mathcal{L}G$, $\mathcal{A} = \Omega^1(S^1,\mathfrak{g})$

(Note: $\mathcal{A}/\mathcal{G} \xrightarrow[\sim]{\text{monodromy}} G/\mathrm{Ad}G \cong T/W$)

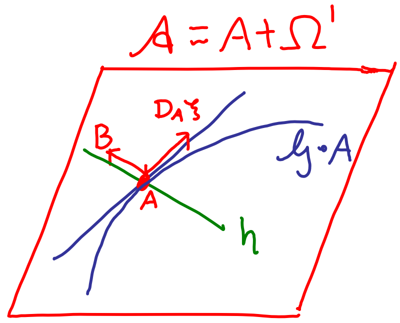
"This behaves like adj. repr. of cpt. Lie groups" (metrics on M $\mathfrak{g}$ $\rightsquigarrow$ metric of $\mathcal{A}$ $\overset{\text{isometry}}{\curvearrowleft}$ $\mathcal{G}$)

$$B(\theta)d\theta,\ C(\theta)d\theta \in T_A\mathcal{A} = \Omega^1(M,\mathfrak{g})$$

$$\Longrightarrow (B, C)_{g_{\mathcal{A}}} = -\int_{S^1} \mathrm{Tr}\, B(\theta)\, C(\theta)\, d\theta$$

Prop. Cartan subalg. $\mathfrak{h} \subset \mathfrak{g} \underset{\substack{\text{const.}\\ \text{1-forms}}}{\hookrightarrow} \Omega^1(M,\mathfrak{g}) = \mathcal{A}$

$\mathcal{G}$-orbit $\perp$ $\mathfrak{h}$ in $\mathcal{A}$



Pf. Given $A \in \mathcal{G}\cdot A \cap \mathfrak{h}$

$\forall B \in \mathfrak{h}$, $\xi \in \Omega^0(S^1,\mathfrak{g})$

$$(D_A\xi, B) \overset{?}{=} 0$$

$$= \int_{S^1} \left\langle \frac{d\xi}{d\theta} + [A,\xi], B \right\rangle d\theta$$

$$= \int_{S^1} \left\langle \cancel{\frac{d\xi}{d\theta}}, B \right\rangle d\theta + \int_{S^1} \underbrace{\langle [A,\xi], B\rangle}_{(\because \mathcal{G} \curvearrowright \mathcal{A} \text{ isometry})} d\theta$$

(by part, $\frac{dB}{d\theta} = 0$)

$$- \langle \xi, \underbrace{[A,B]}_{= 0\ (\because A,B \in \mathfrak{h} \text{ Abelian})} \rangle$$

$$= 0$$

Infinitesimal stabilize $= \mathrm{Ker}\, D_A|_{\Omega^0(S^1,\mathfrak{g})} = ?$

$$0 = D_A\xi = \frac{d\xi}{d\theta} + [A,\xi]$$

i.e. $\xi(\theta) = e^{-\theta\, \mathrm{ad}A}\xi(0)$ and $\xi(1) = \xi(0)$

$$\Longrightarrow \xi(0) \in \mathrm{Ker}(1 - e^{-\mathrm{ad}A})$$

$$\mathrm{Ker}\, D_A|_{\Omega^0(S^1,\mathfrak{g})} = \mathrm{Ker}(1 - e^{-\mathrm{ad}A}) \quad \Longrightarrow$$

$\dim(\text{—— " ——}) = \dim\mathfrak{h} + 2\times(\#\text{ root planes thru. } A \in \mathfrak{h})$

$\mathcal{G}\cdot A$ meets $\mathfrak{h}$ ∞ many times !

Consider $A \in \mathfrak{h}$,

$$A^g = g^{-1}Ag + g^{-1}dg \in \mathcal{G}\cdot A \cap \mathfrak{h}$$

if 1°. When $g \in G \subset \mathcal{L}G$, i.e. const. gauge

$$g^{-1}dg = 0$$

$(\perp \operatorname{Ker} D_A \Longrightarrow)$ $g \in W = \frac{N(T)}{T}$ Weyl group

2° When $g \in \mathcal{L}T \subset \mathcal{L}G$, so $g^{-1}(\theta)\,A\,g^{-1}(\theta) = A$ $\forall\theta$

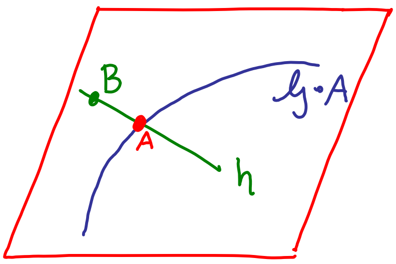
$\Longrightarrow$ $g^{-1}dg \equiv \text{const}$

$\Longrightarrow$ $g(\theta)$ closed 1-parameter subgp. of G

$$g \in \operatorname{Hom}(S,T) \cong \mathbb{Z}^r$$

Fact : No other intersections.

$$\mathcal{G}\cdot A \cap \mathfrak{h} \longleftrightarrow W \ltimes \mathbb{Z}^r$$



Morse theory for distance^2 fu. of $\mathcal{G}\cdot A$ from generic $B \in \mathfrak{h}$

$$E : \mathcal{G}\cdot A \longrightarrow \mathbb{R}$$

$$E(A') = \tfrac{1}{2}(A'-B,\ A'-B)$$

<u>Claim</u>: $\operatorname{Crit}(E) = \mathcal{G}\cdot A \cap \mathfrak{h}$

Pf: $A' \in \operatorname{Crit}(E)$

$\Longleftrightarrow$ $\int \langle \frac{d\xi}{d\theta} + [A',\xi],\ A'-B \rangle = 0$ $\quad\forall\, \xi \in \Omega^0(S^1,\mathfrak{g})$

$\Longleftrightarrow$ $\frac{dA'}{d\theta} + [B,A'] = 0 \Longleftrightarrow A'(\theta) = e^{-\theta\,\mathrm{ad}B}\underbrace{A'(0)}_{A'(1)}$

B generic $\Longrightarrow$ A' const. in $\mathfrak{h}$

§ Moment maps

$$f : M \longrightarrow \mathbb{R}$$

i.e. $S^1 \curvearrowright (M, \omega)$ symplectic

s.t. $\iota_X \omega = -df$, X v.f. gen. by S^1-action.

<u>Prop</u>: f moment map

$\Longrightarrow$ $\mathrm{crit}(f)$ non-degen., $\mathrm{index} f \in 2\mathbb{Z}$

If $\mathrm{crit}(f)$ discrete $\Rightarrow$ f perfect Morse function.

In particular, $H^{odd} = 0$ and $\mathrm{Tor}\, H^\bullet = 0$.

Pf: $p \in \mathrm{Crit}(f) = M^{S^1}$ fix point set.

$S^1 \curvearrowright T_pM$ as representation

Choose an S^1-inv. metric M, so $S^1 \to \mathrm{Isom}(M, g)$

$\left(S^1 \text{ irred rep. : } S^1 \overset{\text{trivial}}{\curvearrowright} \mathbb{R} \text{ or } S^1 \overset{e^{2\pi i n\theta}}{\curvearrowright} \mathbb{R}^2 = \mathbb{C} \longleftrightarrow n \in \mathbb{Z} \setminus 0 \right)$

$$\underset{S^1\text{-action :}}{T_pM} = \underset{\text{trivial}}{N_p} \oplus \underset{\langle n_1, - \rangle}{E_1} \oplus \underset{\cdots\cdots}{\cdots\cdots} \oplus E_N \quad \text{(ortho. dec.)}$$

S^1 acts on geodesics $\Longrightarrow$ $M^{S^1} \overset{\text{loc.}}{\cong} N_p \Longrightarrow M^{S^1}$ smooth

Furthermore: $\mathrm{nbd}(M^{S^1}) \overset{\exp}{\underset{\sim}{\longleftarrow}} \mathrm{Disk}(\tilde{E}_1 \oplus \cdots \oplus \tilde{E}_N)$
$$\downarrow$$
$$M^{S^1}$$

Eg. Coadj. orbit $f : M \subset \mathfrak{g}^* \xrightarrow{\langle X, - \rangle} \mathbb{R}$

(enough to assume X generates a cpt. gp. act^{n} ✓).

Eg. $M = S^2(a) \subset \mathbb{R}^3 = \underline{so}(3)^*$

$x^2+y^2+z^2=a^2$

$$\omega \triangleq x\,dy\,dz - y\,dx\,dz + z\,dx\,dy \in \Omega^2(\mathbb{R}^3)^{SO(3)}$$

$$d\omega = 3\,dx\wedge dy\wedge dz$$

$\rightsquigarrow \omega|_{S^2(a)}$: $SO(3)$-inv. sympl. form.

In general, choose base $x^1, \ldots\ldots, x^m$ for $\mathfrak{g}$

$$[x^i, x^j] = \Sigma\, C_{ijk}\, x^k$$

$\rightsquigarrow x^j : \mathfrak{g}^* \longrightarrow \mathbb{R}$

$$\omega := \tfrac{1}{2}\, C_{ijk}\, x^k\, dx^i \wedge dx^j \in \Omega^2(\mathfrak{g}^*)^G$$

then $\omega|_{\text{Any coadj. orbit}}$: G-inv. symp. form

<u>Claim</u>: $S^1 \leqslant SO(3) \curvearrowright S^2(a)$ rotate about z-axis

$\Longrightarrow$ moment map $f = -a^2 z$

Pf. $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ ($\sim$ rotate z-axis)

$$\iota_X\omega = x^2 dz - xz\,dx + y^2 dz - yz\,dy$$
$$= (x^2+y^2)\,dz - z\,d(x^2+y^2)/2$$
$$= a^2\,dz \qquad (\because x^2+y^2 = a^2 - z^2)$$

§ Exact stationary phase for moment maps

$$f : M^{2n} \longrightarrow \mathbb{R}$$

<u>Prop1</u> Stationary phase approximation.

Hessian

$$\int_M e^{-2\pi i f t}\, d\nu \overset{t\to\infty}{\sim} \sum_{df(p)=0} \frac{1}{t^n} \frac{e^{-2\pi i f(p) t - \frac{\pi}{4} i \,\operatorname{sign} H(f)}}{|\det H(f)_p|^{1/2}}$$

Prop 2 If f is moment map.

$$\int_M e^{-tf}\frac{\omega^n}{n!} = \frac{1}{t^n}\sum_{df(p)=0}\frac{e^{-tf(p)}}{m_p}, \quad \left(m_p = (-1)^{\frac{1}{2}\text{Morse Ind}}\cdot|\det H(f)_p|^{1/2}\right)$$

Eg: $f=-z : S^2 \longrightarrow \mathbb{R}$ (moment map ✓)

$$\frac{1}{4\pi}\int_{S^2} e^{tz}\omega \overset{\text{Prop2}}{=\!=} \frac{1}{t}\left(\frac{e^t}{2}-\frac{e^{-t}}{2}\right) = \frac{\sinh t}{2}$$

Explicit calculation: In cylindrical coord. (r,θ,z)

$$\omega = r^2 d\theta\, dz + z\, r\, dr\, d\theta$$
$$= (r^2+z^2)\, d\theta\, dz = d\theta\, dz \quad (\because r^2+z^2=1 \text{ on } S^2)$$

$X=\frac{\partial}{\partial\theta}$ and $\iota_X\omega = df \Longrightarrow f=-z$

$$\int_{S^2} e^{-tf}\omega = \int_{z=-1}^{1}\int_{\theta=0}^{2\pi} e^{tz}\, d\theta\, dz = 2\pi\frac{1}{t}[e^{tz}]_{z=-1}^{1} = \text{same}$$

Pf. of Prop 1.

Consider distribution $\varphi \in C^\infty(M)$

$$Z(\varphi) := \int_M e^{2\pi i f t}\varphi\frac{\omega^n}{n!}$$

Lemma: $\mathrm{Supp}(\varphi) \cap \mathrm{Crit}(f) = \emptyset$

$\Rightarrow Z(\varphi) \to 0$ faster than any power of t

i.e. $|Z(\varphi)| \leq \frac{C_n(\varphi)}{t^n}$

Pf. of lemma: On $\mathrm{Supp}(\varphi)$, $\exists$ v.f. Y s.t. $2\pi Y(f)=1$

$$0 \underset{(2\pi=1)}{\overset{\text{by part}}{=\!=}} \int \mathcal{L}_Y\left[e^{ift}\varphi\frac{\omega^n}{n!}\right]$$

$$= \int it\underbrace{Y(f)}_{1}e^{ift}\varphi\frac{\omega^n}{n!} + \int e^{ift}\mathcal{L}_Y\varphi\frac{\omega^n}{n!}$$

$$\Longrightarrow |t|\left|\int e^{ift}\varphi\right| \leq \int|\mathcal{L}_Y\varphi| = C$$

indep of t

i.e. $|Z(\varphi)| \leq C/t$

lemma $\Rightarrow$ reduce to local contributions near critical points.

Near critical point $0 \in \mathbb{R}$, $f = \frac{x^2}{2} + \text{h.o.t.}$

- $\int_{\mathbb{R}} e^{-x^2} \frac{dx}{\sqrt{\pi}} = 1$
- $\int_{\mathbb{R}} e^{-ax^2} \frac{dx}{\sqrt{\pi}} = \frac{1}{\sqrt{a}}$ if $\operatorname{Re} a > 0$
- $\int_{\mathbb{R}} e^{-iax^2} \frac{dx}{\sqrt{\pi}} = \frac{1}{\sqrt{a}} e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{a}} e^{-i\frac{\pi}{4}\operatorname{sign}(a)}$
- $Q \in S^2 \mathbb{R}^{n*}$ non-degen.

$$\int_{\mathbb{R}^n} e^{-iQ(x)} \frac{dx^1}{\sqrt{\pi}} \frac{dx^2}{\sqrt{\pi}} \cdots \frac{dx^n}{\sqrt{\pi}} = \frac{1}{|\det Q|^{1/2}} e^{-i\frac{\pi}{4}\operatorname{sign} Q}$$

Hence prop. 1. QED

Proof of prop. 2 (skip).

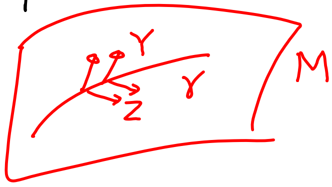
(reason: Equivar. Darboux $\Rightarrow$ $f = ax^2$ exactly wrt std vol.)

Eg. $f = \langle X, - \rangle : \underset{\text{coadj. orbit}}{M} \subset \mathfrak{g}^* \longrightarrow \mathbb{R}$

$$\int_M e^{2\pi i f t} \frac{\omega^n}{n!} \overset{\text{distribut}^{n}}{=\!=} \frac{1}{t^n} \sum^{W} a_p \delta_p .$$

∞ dim "eg." $(M, g) \Rightarrow \mathcal{L}M$ symplectic

$\omega(\gamma)(Y,Z) := \int_{S^1} \langle \nabla_{\dot\gamma} Y, Z \rangle$



$d\omega = 0$, ω (mildly degen.)

$$S^1 \curvearrowright (\mathcal{L}M, \omega) \xrightarrow{E} \mathbb{R}$$

moment map is $E(\gamma) = \frac{1}{2} \int |d\gamma|^2$ energy.

$(\mathcal{L}M)^{S^1} = M$ const. loops.

$$\int_{\mathcal{L}M} e^{2\pi i t E} \frac{\omega^{n=\infty}}{n!} \text{ “}=\text{” } \int_M \hat{A}(M) = \operatorname{Index} \underset{\text{Dirac operator}}{\not{D}}$$

§ Atiyah-Bott's Lefschetz fix pt. formula

$$\underset{\text{sympl. mfd}}{(M,\omega)} \rightsquigarrow \xrightarrow[\text{(need } [\omega]/\mathbb{Z})]{\text{Quantization}} \underset{\text{Hilbert space}}{\mathcal{H}}$$

e.g. $\mathcal{H} = H^0_{\bar\partial}(M, L)$ w/ $c_1(L) = [\omega]$
(use complex polarization).

$$G \curvearrowright (M,\omega) \Longrightarrow G \curvearrowright \mathcal{H}$$

Conversely, representations should be realized as quantization of coadjoint orbits.

Eg. Representations of $G = SU(2)$

Std. repr. $SU(2) \curvearrowright \mathbb{C}^2 = V$

$\{S^n V\}_{n=1}^{\infty}$ is a complete list of irred. G-repr.

Compact group $\sim$ Finite group

Repr. $G \overset{\rho}{\curvearrowright} V$ is determined by character

$\chi_V : G \longrightarrow \mathbb{C}$, $\chi_V(g) = \mathrm{Tr}_V\, \rho(g)$ (class fu.).

Again, determined by $\chi_V|_T$ (by conjugtion thm).

So $\iota_T^* : \mathrm{Rep}\, G \hookrightarrow \mathrm{Rep}\, T$.

For $\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix} \in T \leq SU(2) \curvearrowright V = \mathbb{C}^2$ std.

$\chi_V\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix} = z + z^{-1}$. Write $\iota_T^* V = L \oplus L^{-1}$

$\Rightarrow \iota_T^*(S^2 V) = S^2(L \oplus L^{-1}) = L^2 + \mathbb{C} + L^{-2}$

$\chi_{S^2V} = z^2 + 1 + z^{-2}$

Similarly, $\chi_{S^nV} = z^n + z^{n-2} + \cdots\cdots + z^{-n}$

How do these come from quantization?

Coadj. orbits of $SU(2) \longleftrightarrow S^2(a) \subset \mathbb{R}^3 = \underline{su}(2)$ w/ $a \geq 0$

$$[\omega|_{S^2(a)}] = a \in H^2(S^2(a)) = \mathbb{R}$$

quantized condition: $a = n \in \mathbb{Z}_{\geq 0}$

$$\Longrightarrow \exists L \text{ s.t. } c_1(L) = [\omega|_{S^2(n)}], \quad \begin{tikzcd} & L \arrow[d] \\ SU(2) \arrow[ur, bend left] \arrow[r, bend left] & S^2(n) \end{tikzcd}$$

Choose compat. cpx. str. (i.e. cpx. polarization)

$$\Longrightarrow (S^2(n), L) = (\mathbb{CP}^1, \mathcal{O}(n)) \quad \begin{pmatrix} \text{unless} \\ n = 0 \end{pmatrix}$$

$$\rightsquigarrow SU(2) \curvearrowright \mathcal{H} = H^0_{\bar\partial}(\mathbb{CP}^1, \mathcal{O}(n)) = S^n V$$

$$H^{>0}_{\bar\partial}(\text{——} \,''\, \text{——}) = 0$$

(Should really use virtual repr. $\sum_i (-1)^i H^i(M, L)$)

Note: $\begin{tikzcd} & L \arrow[d] \\ G \arrow[ur, bend left] \arrow[r, bend left] & M \end{tikzcd} \rightsquigarrow G \overset{\rho}{\curvearrowright} H^\bullet(M, L)$

$\rightsquigarrow$ character $\chi : G \longrightarrow \mathbb{R}$

$$\chi(g) = \sum_i (-1)^i \operatorname{Tr}_{H^i} \rho(g)$$

<u>Claim:</u> χ : can be computed by fix pt. formula.

In general, $\forall$ holo. $\begin{tikzcd} L \arrow[r, "\hat\varphi"] \arrow[d] & L \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd}$ (Ω)

$$\rightsquigarrow \varphi^* : H^i(M, L) \longrightarrow H^i(M, L)$$

Lefschetz number

$$L(\varphi) \triangleq \sum (-1)^i \operatorname{Tr}_{H^i} \varphi^* \overset{\text{Thm}}{=\!=} \sum_{\varphi(p) = p} \frac{\operatorname{Tr} \hat\varphi(p)}{\det(1 - \partial\varphi_p)}$$

Recall $\varphi(p) = p \Longrightarrow \hat\varphi(p) : L_p \longrightarrow L_{\varphi(p) = p} \Longrightarrow \operatorname{Tr}$ ✓

$\Downarrow$

$$\partial\varphi_p : T^{1,0}_p M \longrightarrow T^{1,0}_p M$$

Recall: Classical Lefschetz fix point formula.

$$\varphi : M \longrightarrow M \quad \text{s.t. } \mathrm{Graph}(\varphi) \pitchfork \Delta \subset M \times M$$

$$L(\varphi) \triangleq \#\text{fix pt.} = \sum_{\varphi(p)=p} (\pm 1) = \# \, \mathrm{Graph}(\varphi) \cap \Delta$$

Thm. $L(\varphi) = \sum_i (-1)^i \operatorname{Tr} \varphi^* \big|_{H^i(M,\mathbb{R})}$

In the special case $\varphi = 1_M$

$$\#\text{fix pt} = \# M = \chi(M) = \sum (-1)^i b_i = \sum (-1)^i \operatorname{Tr} 1 \big|_{H^i}$$

"Reason": 1° Finite set M

$$\varphi : M \longrightarrow M \rightsquigarrow \varphi^* : C(M) \longrightarrow C(M) \left(= \Omega^\bullet(M)\right)$$

basis δ_{x_i}'s $x_i \in X$

$$\Longrightarrow \operatorname{Tr} \varphi^* = \# \mathrm{Fix}(\varphi)$$

Note: $\varphi^* f(x) = \sum_{y \in X} \underbrace{\delta(y - \varphi(x))}_{K(x,y)} f(y)$ matrix/kernel

$$\operatorname{Tr} \varphi^* = \sum_{x \in X} K(x,x)$$

2° Smooth manifold M

On $\Omega^0(M)$, as above

$$\begin{aligned} \operatorname{Tr}_{\Omega^0} \varphi^* &= \int_X K(x,x)\, dx \\ &= \int_X \delta(\underbrace{x - \varphi(x)}_{u})\, dx \qquad (du = (I - d\varphi)\, dx) \\ &= \int_X \delta(u) \frac{du}{1 - d\varphi} \simeq \sum_{u \in \mathrm{Fix}(\varphi)} \frac{1}{|1 - d\varphi(u)|} \end{aligned}$$

Similarly,

$$\operatorname{Tr}_{\Omega^i} \varphi^* = \sum_{u \in \mathrm{Fix}(\varphi)} \frac{\operatorname{Tr} \varphi^* \big|_{\Lambda^i T_u^*}}{|\det(1 - d\varphi(u))|}$$

Finite dim. cpx. $(\Omega^\bullet, d) \circlearrowleft \varphi^*$ chain map

$\Longrightarrow \quad \mathrm{Tr}\, \varphi^*|_{\Omega^\bullet} = \mathrm{Tr}\, \varphi^*|_{H^\bullet}$

Back to manifold case,

$$\mathrm{Tr}\, \varphi^*|_{H^\bullet} = \mathrm{Tr}\, \varphi^*_{\Omega^\bullet}$$

$$= \sum_{u \in \mathrm{Fix}(\varphi)} \frac{\sum (-1)^i \,\mathrm{Tr}\, \Lambda^i \dot{\varphi}(u)}{|\det(1 + \dot{\varphi}(u))|}$$

$$= \sum_{\mathrm{Fix}(\varphi)} \frac{\det(1 + \dot{\varphi}(u))}{|\det(1+\dot{\varphi}(u))|} = \sum_{\mathrm{Fix}(\varphi)} (\pm 1)$$

$$\left(\mathrm{Det}(I - A) = \sum (-1)^i \,\mathrm{Tr}\, \Lambda^i A \right)$$

$$\left(\text{If } A \in O(2n) \Longrightarrow \sqrt{\mathrm{Det}(I-A)} = \Delta^+(A) - \Delta^-(A) \right)$$

3° Holomorphic setting

$$\begin{tikzcd} E \arrow[r, "\hat{\varphi}"] \arrow[d, "\mathbb{C}^r"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd}$$

$$L(\varphi) \triangleq \mathrm{Tr}\, \varphi^*|_{H^{0,\bullet}(M,E)}$$

$$\text{“}=\text{”}\ \mathrm{Tr}\, \varphi^*|_{\Omega^{0,\bullet}(M,E)}$$

$$= \sum_{\mathrm{Fix}(\varphi)} \frac{\cancel{\left(\sum_i (-1)^i \,\mathrm{Tr}\, \Lambda^i \varphi''\right)}\, \mathrm{Tr}_E\, \hat{\varphi}^*}{\det(1 - \varphi') \cdot \cancel{\det(1 - \varphi'')}}$$

$$= \sum_{\mathrm{Fix}(\varphi)} \frac{\mathrm{Tr}_E\, \hat{\varphi}^*}{\det(1 - \varphi')}$$

4° Elliptic complex $\quad 0 \to \Gamma(E_0) \xrightarrow{D} \Gamma(E_1) \xrightarrow{D} \cdots\cdots$

$D^2 = 0$, symbol seq. is exact.

(Atiyah-Bott fix point formula)

$$\mathrm{Tr}\, \varphi^*|_{H^\bullet(E_\bullet)} = \sum_{\mathrm{Fix}(\varphi)} \frac{\sum_i (-1)^i \,\mathrm{Tr}\, \varphi|_{E_i}}{|\det(1 - \dot{\varphi})|}$$

Back to $G \curvearrowright \underset{\text{coadj. orbit}}{M} \subset \mathfrak{g}^* \rightsquigarrow G \curvearrowright \mathcal{H} = H^\bullet(M, L)$

$\chi : T \leqslant G \longrightarrow \mathbb{C}$?

For simplicity, $M = G/T$. (i.e. generic)

Claim: Generic $t \in T \rightsquigarrow \varphi_t : G/T \xrightarrow{L_t} G/T$

$$\mathrm{Fix}(\varphi_t) \simeq W$$

Pf: $\varphi_t([g]) = [g] \iff tgT = gT$

$\iff g^{-1} t g \in T \iff g^{-1} T g \in T$ ($\because t$ generic)

$\iff [g] \in \frac{N(T)}{T} = W$

Eg. $z = e^{2\pi i \theta} \in S^1 \simeq T \leqslant SU(2) \curvearrowright S^n V = H^0(\mathbb{CP}^1, \mathcal{O}(n))$

$$\chi(z) \underset{\text{formula}}{\overset{\text{Fix pt.}}{=\!=}} \sum_{w \in W} \left(\frac{z^n}{1 - z^{-2}} \right) \cdot w = \frac{z^n}{1 - z^{-2}} + \frac{z^{-n}}{1 - z^2} = \frac{z^n - z^{-n-2}}{1 - z^{-2}}$$

$$= z^n + z^{n-2} + \dots\dots + z^{-n}$$

In general, $\lambda : T \longrightarrow \mathbb{C}^\times$

$$\chi(\lambda) \underset{\text{formula}}{\overset{\text{Fix pt.}}{=\!=}} \sum_{w \in W} \left(\frac{\lambda}{\prod (1 - \alpha)} \right) \cdot w \quad \left(\begin{matrix} \alpha : T \to \mathbb{C}^\times \\ \text{negative root} \end{matrix} \right)$$

$\rho := \prod_{\alpha \in R_+} \alpha$ has a square root (i.e. G/T is Spin)

$$\chi(\lambda) = \sum_W \frac{\lambda \sqrt{\rho}}{\prod (1 - \alpha) (\prod \alpha^{-1})^{1/2}} \cdot w = \frac{1}{\prod (\alpha^{-\frac{1}{2}} - \alpha^{\frac{1}{2}})} \sum_W (-1)^{\operatorname{sgn} w} (\lambda \sqrt{\rho})^w$$

i.e. Weyl character formula.